Objective D (Rate of Return, Risk Loads, and Contingency Provision)

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SUMMARY OF READINGS

- **Ferrari (The Relationship of UW, Investment, Leverage, and Exposure to Total Return on Owners' Equity)**

  Ferrari's paper discusses how calculate total return on equity (capital), breaking the formula for return on equity into several pieces to illustrate how adjusting various ratios can affect profit.

- **McClenahan (Insurance Profitability)**

  McClenahan presents an argument for why measuring insurance "profit" looks very different to a regulator versus a consumer versus an investor.

- **Feldblum (IRR)**

  This paper introduces the concept of the **internal rate of return**, a common method used to determine the profit of an insurance business. This is a simplified look at the IRR calculations shown in the Robbin paper.

- **Robbin (The Underwriting Profit Provision & IRR, ROE, and PVI/PVE)**

  The two Robbin papers discuss several methods of determining an appropriate profit provision for a book of business.

- **Kreps (Riskiness Leverage Models)**

  Kreps discusses a fairly mathematically-intense method of capital allocation.

- **Mango (An Application of Game Theory: Property Catastrophe Risk Load)**

  The Mango paper presents another method of capital allocation.

Ferrari (The Relationship of Underwriting, Investment, Leverage, and Exposure to Total Return on Owners' Equity)

Total Return on Equity – The Basic Equation

As is the case in many businesses, insurance companies operate with a leveraged capital structure. Unlike other enterprises though, insurance leverage does not result from debt capital, but instead from a so-called "insurance leverage," resulting from the deferred nature of insurance liabilities. Ferrari uses the concept of insurance leverage to create a relationship between return on assets and return on equity.

The paper defines the variables:

T = total after-tax return to insurer

I = Investment gain or loss (post-tax)

U = underwriting profit or loss (post-tax)

P = premium income

A = total assets

R = reserves and other liabilities (except equity in unearned premium reserves)

S = stockholders' equity (capital, surplus and equity in unearned premium reserve)

By definition we have: $T = I + U$, $A = R + S$, and total return on equity, which is our item of concern, is just $\frac{T}{S}$.

Through a little algebra, the authors derive their basic equation for return on equity:

$$\text{Total Return on Equity} = \frac{T}{S} = \frac{I}{A}\left(1 + \frac{R}{S}\right) + \frac{U}{P} \cdot \frac{P}{S}$$

The manipulation of the variables is chosen in such a way that the ratios reflect relevant insurance information. The formula shows that return on equity is a function of **investment return on assets $\frac{I}{A}$**, **insurance leverage $\frac{R}{S}$** (called a "factor" in the paper to reference the entire parenthetical relationship **$1 + \frac{R}{S}$**), **underwriting profit $\frac{U}{P}$** and **insurance exposure $\frac{P}{S}$**. Stuff that is nice about this formulation includes:

- It contains the underwriting profit ratio, which is often used for measuring underwriting results.
- It combines the relationship between return on equity (investors' viewpoint), return on assets (society's viewpoint), and return on sales (regulators'/ actuaries' viewpoint).
- It contains the insurance exposure ratio P÷S, which is sometimes used as a quick indicator of insolvency risk.
- It makes it clear that P/S and U/P contribute to return on equity in a comparable way as sales margins multiplied by turnover rates as applied to manufacturing returns.

Ferrari (The Relationship of Underwriting, Investment, Leverage, and Exposure to Total Return on Owners' Equity)

The formula can be alternately expressed as

$$\frac{T}{S} = \frac{I}{A} + \frac{R}{S}\left(\frac{I}{A} + \frac{U}{R}\right)$$

In this formulation:

- R is viewed as "reserve capital," the amount of total investable assets supplied by other than owners.
- The leverage factor R/S is applied to both interest income on assets and underwriting profit as a ratio of reserve capital. Here underwriting losses can be considered interest paid for the use of reserve capital R. A key difference is that in debt capital the level of debt is fixed, but in this formulation, the debt is variable.
- This form makes it clear that underwriting losses can be sustained as long as I/A is high enough to offset them.

Example (Table 1 from paper)

Four companies have operating results as below:

	Company A	Company B	Company C	Company D
Invested Assets (A)	\$20	\$20	\$20	\$20
Reserve Liabilities (R)	\$0	\$6.67	\$10	\$13.33
Owners' Equity (S)	\$20	\$13.33	\$10	\$6.67
Investment Return (I/A)	5%	5%	5%	5%
Leverage Ratio (R/S)	0	0.5	1.0	2

Company A is an unlevered investment trust. Companies B, C, and D have low, medium and high leverage, respectively.

- Note that invested assets = Reserves + Equity, as noted above (A = R + S)
- Leverage ratio is reserves ÷ equity

We look at the return on equity in three different scenarios for underwriting profit (U/R) – profit of 6%, breakeven, and loss of 6%

	Company A	Company B	Company C	Company D
U/R = +6%	5.0 %	10.5 %	16 %	27 %
U/R = 0%	5.0	7.5	10	15
U/R = -6%	5.0	4.5	4	3

Sample Calcs; using $\frac{T}{S} = \frac{I}{A} + \frac{R}{S}\left(\frac{I}{A} + \frac{U}{R}\right)$

- Company B, breakeven: $\frac{T}{S} = 5\% + 0.5(5\% + 0\%) = 7.5\%$
- Company C, +6%: $\frac{T}{S} = 5\% + 1.0(5\% + 6\%) = 16\%$

Note the relative variability of returns in each different leverage situation – Company D is very leveraged and allows for the greatest return if underwriting results are good, but the least return if they are not so hot.

Implications on the Optimal Capital Structure

If we accept the view of reserves as leverage-inducing non-equity capital, we should also consider how to determine the optimal capital structure – the composition of liabilities and owners' equity to maximize firm value (or, alternatively the optimal reserve:surplus ratio). To do so, we consider the influences from the expected earnings stream and the rate at which earnings are capitalized by the market.

Reserves will add to the income stream as long as the costs of financing reserves are offset by investment income. Thus we must consider the effect of reserves on the stability of earnings. This is where actuarial work comes into play. When we consider probability of ruin or insolvency, we should consider the likelihood of unfavorable returns to owners and potential loss of equity capital. The optimal reserve position will depend on the investment policy.

There are several reasons that it is important to analyze the optimal capital structure:

- If the public views the industry as having a capacity problem, it could be that the investor demands a relatively low leverage ratio.
- A capacity problem may be attributable in part to aggressive investment portfolios that force the optimal capital structure to a low leverage ratio.
- Alternatively, if the optimal structure demands a higher leverage ratio than currently held, we could view the industry as being over capitalized.

Balcarek's Discussion

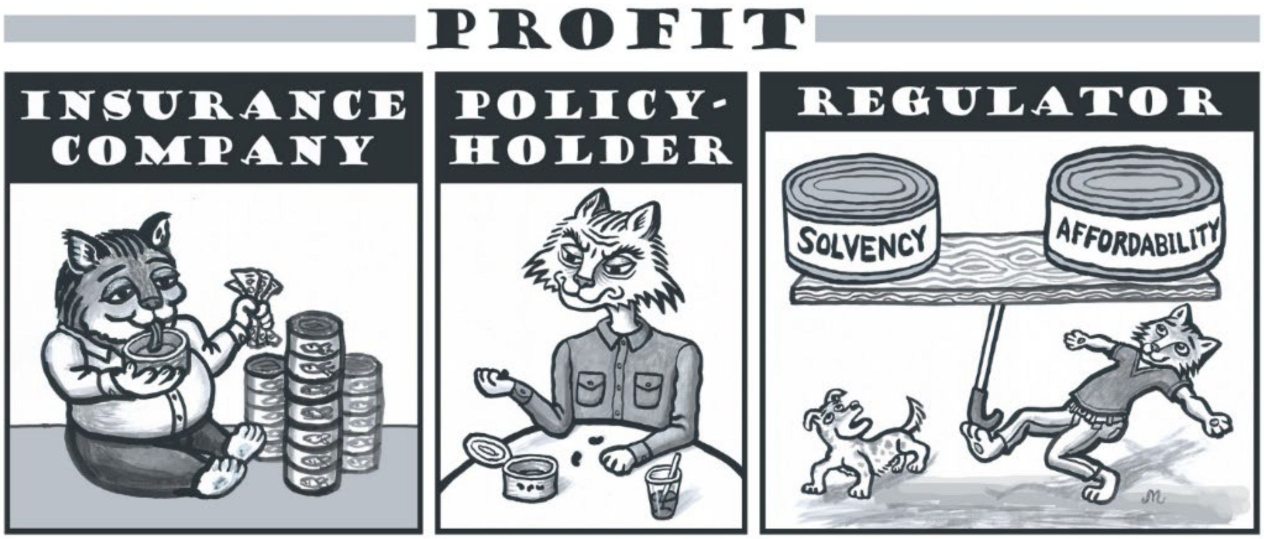
The comments in the discussion portion are really quite interesting. The discussion suggests using the formulas with caution:

- Although the second term in the formula, $\frac{U}{P} \cdot \frac{P}{S}$, suggests that increasing premium volume will necessarily increase return on equity, Balcarek cautions that the equity would be exposed to more risk, which should be considered.

- The relationships suggested by the formulas are likely cross-offsetting. For example:
  - If insurance exposure (premium: surplus) increases, then underwriting profit may decrease (additional business written may be of poorer quality). *P/S and U/P move in opposite directions*.
  - When insurance exposure increases, investment gain on assets will decrease because there will be an increase in uninvested assets like agents' balances, likely forcing the company to adopt a more conservative investment strategy to offset the riskiness of a higher insurance exposure level. *(P/S and I/A move in opposite directions.*)
  - If the insurer has favorable underwriting profit results (high U/P), it can write a larger volume of business (high P/S). This would indicate that *U/P and P/S move in the same direction*. And yes, this directly contradicts the first bullet, but a takeaway is that the same forces could potentially have counteracting effects.
  - Favorable underwriting profit would allow the insurer to take advantage of riskier assets, and earn a higher investment return (*U/P and I/A move in the same direction*). Again you might note that this point directly contradicts points 2 and 3, but the idea is that either theory could be seen as plausible.

McClenahan (Insurance Profitability)

The connotation of insurance profitability is dependent upon the audience. For investors and (non-ACA health) insurers, it's pretty awesome, while to policyholders, not so much. To holders of mutual company insurance policies, there is no connotation, and to regulators it is dependent on balancing solvency and affordability of rates. This paper examines how to measure profit in the insurance world.



While the concept of measuring profit sounds pretty straightforward, in insurance, it's fairly complex. In most businesses, it is clear how much money was spent and earned at the end of a given period. For insurance, it can be several years before losses fully emerge. The need to hold reserves not only based on current policies but also due to past reserving decisions adds an additional level of complexity to the problem of measuring insurance profitability.

Profit vs. Rate of Return

Profit is not rate-of-return (ROR). Here is a fun table to summarize their differences:

Profit	**Rate-of-Return**
Excess of revenues over expenditures	Ratio of profit to equity (or assets, sales, etc.)
Monetary value; absolute meaning in currency	Efficiency measure; meaning relative to alternative RORs
Important to investors and management as source of dividends and growth	Important to prospective investors for purposes of comparing alternate investment choices
Provides security against insolvency	

Profit – Ratemaking Basis

Insurers generally gain profit from distinct two sources – underwriting (actual insurance risk) and investment (from investment of policyholder-supplied funds). On top of that, the insurer may gain investment income by directly investing funds from shareholders. Although these are distinct sources, rate regulation essentially boxes insurers into arbitrarily distinguishing between investment income from policyholder funds versus that coming from shareholder funds.

Insurance differs from many other services in that the insureds pay upfront for an event that is uncertain, as opposed to paying for a good or service that has, or will have been received, at a certain time in a known quantity. This results in an opportunity cost to the insured, who could simply invest the premium himself until it is actually needed to pay a claim.

McClenahan posits that this opportunity cost should be calculated on the cash flows associated with the line of business, including overhead costs. Moreover, **the calculation should be made at a risk-free rate of return**. The reasoning is that a policyholder does not receive dividends based on the performance of the company, nor does he bear the burden of poor investment returns, so any investment income (or loss) should not be considered "owned" by the policyholder in the ratemaking process.

Lastly, McClenahan argues that investment income on surplus should also be excluded from ratemaking, for the same reason. It *protects* the policyholder but is not *owed* to him. If a company has a large surplus and incorporates that into the rates, the policyholder would see lower rates because there is less risk of insolvency. By contrast, if a company has a small surplus, the policyholder would see higher rates for a policy that is less stable, which is counterintuitive.

Illustrative Example:

We have two insurers with equal loss exposures.

- Expected Losses = 65,000
- Expenses = 35,000
- Risk-free rate of return = 5%

Premium is collected at the beginning of the year; expenses are paid at the beginning of the year; losses are paid at the end of the year.

Both insurers look to break even (considering all sources of profit) at the end of the year. Insurer A has $500,000 in contributed capital and surplus. Insurer B has $0.

What should each insurer charge?

Insurer A: Income will be sum of underwriting profit and investment income. We want:

$$(P - 65 - 35) + (500 + P - 35)(0.05) = 0 \rightarrow P = 73{,}095$$

Insurer B: $(P - 65 - 35) + (P - 35)(0.05) = 0 \rightarrow P = 96{,}905$

Insurer A will charge less because they can use funds from contributed capital to supplement income. This shouldn't be – both insurers have the same loss exposure; they should in theory charge the same amount – policyholders from A shouldn't receive additional benefit from stakeholders subsidizing their costs.

Rate of Return – The Appropriate Denominator

When determining a desired rate of return, two common choices for the denominator are sales and equity. Equity arises as an appropriate basis with which to measure company-wide financial performance relative to other companies, but McClenahan points out two issues with using equity as a basis for regulation:

- **Return on equity forces the regulator to forego rate equity for rate-of-return equity**. In other words, companies that have identical risks, but different degrees of leverage (Premium ÷ Surplus, which Ferrari calls *insurance exposure*), would require different rates to be considered equal under the return on equity measure.
- Requires an allocation of equity by line of business and state. We see throughout this syllabus that *allocation* of equity is largely arbitrary and artificial, as the entire surplus can be used for any risk.

Both of these issues could be overstepped by assigning some fixed value to be used as a leverage factor, but then the regulators are no longer regulating return-on-equity, but return-on-sales, as measured through a somewhat convoluted process.

The other natural choice of denominator – sales – relates the profit provision in the premium to the premium itself. It is intuitively related to "markup," and is therefore meaningful and useful to the consumer. The application can be done in any number of ways (such as setting a return-on-equity provision), but regardless of the method chosen, the application is premium-based, and does not consider equity. This is what McClenahan deems to be true rate regulation.

Profitability Standards

The regulator is tasked with determining what constitutes "reasonable, not excessive, not inadequate" profit provisions within rates. Because profit comprises a relatively small portion of an insurer's total cash flows, any over- or underestimation of liabilities, the most volatile and substantial component of the cash flows, has an enormous effect on profitability.

The Statement of Principles stipulates that "the underwriting profit and contingency provisions are the amounts that, when considered with net investment and other income, provide an appropriate total after-tax return." While this statement has the benefit in allowing for flexibility of return levels, it also does allow that some jurisdictions may deem excessive a profit margin that is acceptable in another.

The choice of what rate is deemed reasonable will work jointly with the market conditions. If insurers deem regulatory conditions to be inappropriately restrictive, this will affect the number of insurers available and the level of underwriting intensity used.

Illustrative Example: An insurer is setting rates for a certain line of business, subject to the following simplified assumptions:

- Loss = 100
- Fixed Expenses = 25
- Variable Expenses, as a portion of premium = 20%
- Profit = 5%
- Ignore the time value of money

The indicated premium in this case would be $\frac{25+100}{1-0.20-0.05} = 167$.

If all goes as planned, then the insurer's profit would be 5%.

If losses are 10% higher than expected, the insurer's profit would be $\frac{0.8(167)-25-110}{167} = -1\%$ (1.2 times lower than expected), demonstrating the sensitivity of profit to loss assumptions.

Feldblum (IRR)

Background

This paper delves into an insurance application of Internal Rate of Return (IRR). It assumes a basic understanding of what is IRR, which might be familiar to you (at least at a basic level) from VEE stuff, or from your routine lively bar chats about different types of financial measures.

Introduction

Early rating bureau pricing procedures incorporated a fixed underwriting profit margin, but the lack of theoretical justification, volatile interest rates, and increasing competitiveness stimulated an interest in more refined pricing models. Specifically:

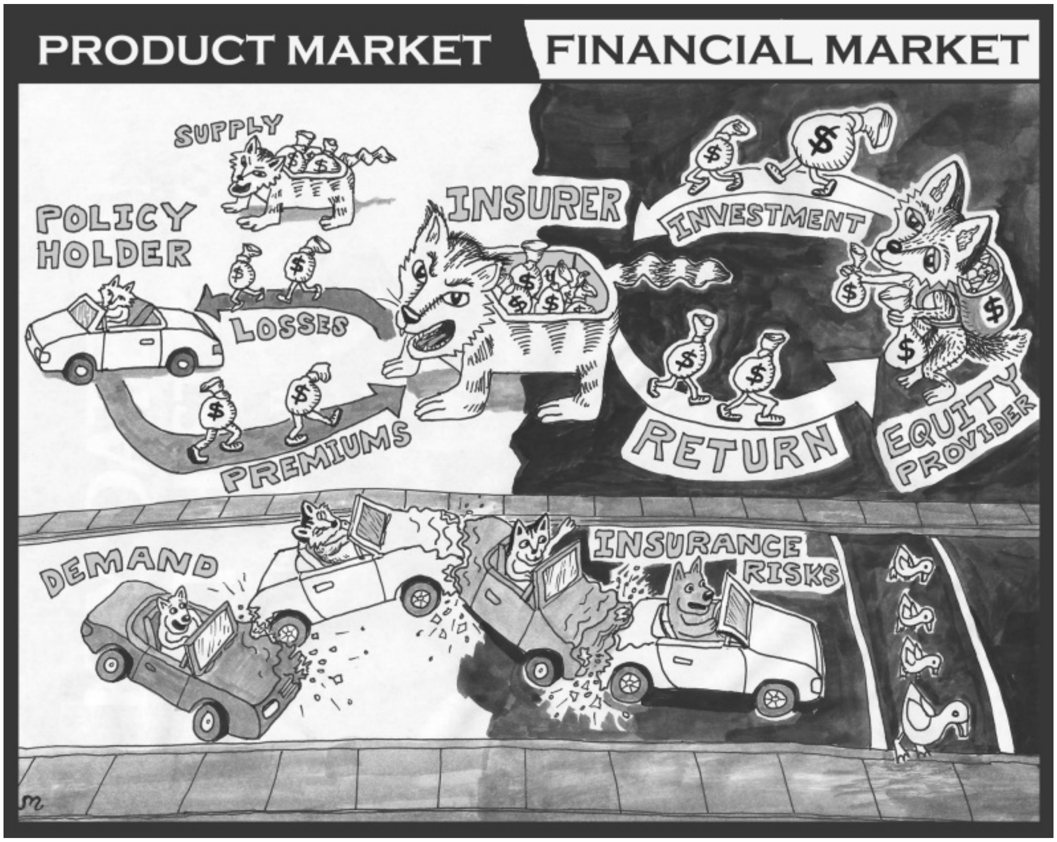
- Time value of money: Because insurance inflows (premiums) and outflows (claims) occur at different times, the value of cash flows can be significantly different depending on timing and interest rates.
- Competition and expected returns: Insurance is unlike other goods in that the cost is unknown prior to setting prices, so insurers desire a pricing model refined enough to be competitive.
- The rate base: Underwriting profit margin is a return on sales but return on assets or equity might be more appropriate.

Points of View

Insurance transactions may be evaluated from two points of view:

- Insurer ↔ Policyholder relationship: Policyholder pays premiums for insurance to compensate covered losses. Both transactions occur in the *product market*, influenced by supply and demand.
- Equity Provider ↔ Insurer Relationship: Shareholders invest funds in company, which provides a return. Both transactions occur in the *financial market*, and expected returns are influenced by insurance risks.

The views are related in that the supply of insurance in the product market depends on the cost of capital, and expected returns are dependent on consumers' demand for insurance.



Simple IRR Example (Exhibit 1 in Text)

Suppose a firm's cash flows for a given venture in years 0, 1, and 2 are given by $\{-100{,}000,\ 65{,}000,\ 65{,}000\}$, respectively. The IRR (internal rate of return) is given by:

$$0 = -100{,}000 + \frac{65{,}000}{1 + IRR} + \frac{65{,}000}{(1 + IRR)^2}$$

Solving, we get IRR = 19.5%. In words: the IRR model suggests that the firm should accept the venture as long as the cost of capital is less than 19.5%.

The difference between the above example and the cash flows of an insurance operation is that in the above, there was an initial outlay of funds (-100,000), while in insurance applications (as per the insurer↔ policyholder relationship), the signs are opposite of that – there is an initial influx of funds (premiums) and then negative flows later on as claims are paid.

Looking at it from the equity perspective however, the signs would be similar to those of the example: we do have an outflow initially (investments in securities to support UPR and loss reserves) and an inflow as losses are paid, allowing supporting capital to be released.

Equity Flows

Feldblum argues that the IRR model should focus not on the Insurer ↔ Policyholder relationship, but on the Equity Provider ↔ Insurer relationship. The IRR pricing model in this paper takes the viewpoint of the equityholders, who are concerned with insurance transactions since they impact the amount of surplus needed. The analysis from this viewpoint is similar to that of capital budgeting – firms use IRR analyses to value investments that require an initial outlay of capital and then returns in future years.

The amount of surplus required can be deduced from the actual industry surplus held, assuming efficient markets. (If the industry were overcapitalized, investors would withdraw funds, but invest more if the industry were undercapitalized.)

While the IRR model makes use of surplus, it does not necessitate the assumption of efficient markets, since surplus can be determined in other ways. But you should know that surplus has a large impact on the results of an IRR model, both in terms of the amount of surplus as well as the timing of its release. We saw in other readings how the amount can be determined in various ways. (Even assuming a fixed and known company surplus, allocation by lines of business is largely arbitrary.) Additionally, the IRR model requires assumptions regarding the timing of surplus release.

Simple IRR Example 2 (Equity Flow Illustration in the Text)

Using some simplified assumptions about an insurer, we can set up an IRR model to determine required return. The simple example in the text implicitly assumes that cash flows are incurred as they are paid so does not differentiate between GAAP and STAT flows. (The example in the appendix differentiates between incurred and paid cash flows, but I would recommend the Robbin IRR exhibit for a detailed calculation showing this difference, since it is a little easier to follow.)

- 1,000 of premium on January 1 of Year 0
- 500 in claims paid on January 1 of Year 1, and 500 paid on January 1 of Year 2
- Management requires a 2:1 ratio of undiscounted reserves:surplus
- 10% return on assets
- Reserves and surplus are investible assets

We determine the equity flows (amounts paid by and to shareholders) as follows:

Year 0 Equity Flows

- Insurer receives 1,000 in premiums
- Undiscounted reserves are 1,000 so at a 2:1 ratio, required surplus of 500 is raised from equity holders. This is an equity flow of **-500.**
- Insurer invests assets of 1,000 + 500 at 10%, earning 150 for a total of 1,650 at end of year.

Year 1 Equity Flows

- Insurer starts off with assets of 1,650
- Insurer pays 500 in claims, leaving 1,150 in assets.
- Undiscounted reserves are now 500, requiring a surplus of 250 (so 750 must be held).
- The excess between assets and required reserves and surplus is returned to equityholders. Equity flow of 1,150 – 750 = **400.**
- After returning surplus to equityholders, insurer is left with 1,150 – 400 = 750 in assets. After a 10% return, this is 825.

Year 2 Equity Flows

- Insurer has assets of 825.
- Insurer pays 500 in claims, leaving 325 in assets.
- Undiscounted reserves are now 0, requiring 0 surplus, so all assets are returned to shareholders, giving an equity flow of **325.**

Our IRR equation is thus solved with:

$$0 = -500 + \frac{400}{1 + IRR} + \frac{325}{(1 + IRR)^2}; \ \ IRR = 30\%$$

In general, to determine equity flows for each year, consider something like the table below. Equity flows are the shortfall between required assets and actual cash.

		Year 0	Year 1	Year 2	...	Year n
Cash	BOY Assets	0				
	+ Premium					
	- Paid Loss & Expense					
	Total Cash					
Assets	Reserves					0
	Surplus (based on ratio)					0
	Total Assets					0
	EOY Assets (after inv. Income)					0
	Equity Flow (Cash – Assets)					

For the last example, the equity flows in this table format would look like:

		Year 0	Year 1	Year 2
Cash	BOY Assets	0	1,650	825
	+ Premium	1000		
	- Paid Loss & Expense		-500	-500
	Total Cash	1,000	1,150	325
Assets	Reserves	1,000	500	0
	Surplus (based on ratio)	500	250	0
	Total Assets	1,500	750	0
	EOY Assets (after inv. Income)	1,650	825	0
	Equity Flow (Cash – Assets)	-500	+400	+325

Generally, you can use this method for deriving IRR equity flows for more "streamlined" questions – where there is no differentiation between GAAP and STAT cash, nor between paid and earned cash. For more complex problems, you should use a more extended example, described more in the Robbin IRR paper.

Surplus

Surplus is meant to protect the insurer against any number of risks, including:

Occur During Policy Period	Until All Losses are Paid
• Pricing risk (adverse experience) • Catastrophe risk (e.g., hurricanes, earthquakes)	• Asset risk (financial assets losing value) • Reserving risk • Asset-liability mismatch (changes in interest rates affect assets and liabilities differently) • Credit risk (e.g., failing to collect premiums or reinsurance)

While the idea of surplus is quite important, surplus itself is a largely theoretical concept that cannot be directly measured. This creates some difficulty with the use of IRR in insurance, as compared with its use for firms dealing with concrete commodities.

Firstly, it is difficult to compare IRR across insurance firms, as the amount of allocated surplus need not be similar, even when comparing identical lines of business. Secondly, while the IRR model focuses on the capital market (cash flows to and from investors), insurance contracts are priced based on the product market (supply and demand).

At a macro level, this is largely nonconsequential as the two markets would tend toward equilibrium, but at a firm level, this can create a major discrepancy. (To remedy this issue, some analysts use industry surplus levels, rather than those of an individual firm.)

Surplus is generally allocated using some base (e.g., premium or reserves). If you've not yet blotted out the "Exam 6 experience" from your memory, you may recall that one of the tests of surplus adequacy is the ratio of written premium to policyholder surplus. This test tends to imply that surplus should vary directly with premium, and that surplus is committed when the premium is written and released on expiry.

Another test compares surplus to reserves held. This test would imply that surplus varies with reserves, and that surplus is committed when losses occur, and released when paid (or, if the loss portion of the unearned premium reserve is included on top of the loss reserve, then surplus is committed when policy is written, and released when losses are paid). The use of reserves requires consideration as to what reserves should be used (L&L reserves only, or also UPR?).

Each of these has different implications that may or may not be appropriate, but in general, some things to consider in surplus allocation are described in the text:

- Long- versus short-tailed lines: When surplus is allocated based on reserves, longer-tailed lines are allocated relatively more.

> Using an example similar to that in the text, assume the following:
>
> - An insurer writes business in two lines, Homeowners' Insurance (HO) and Workers' Compensation (WC).
> - Each line writes 100 in premium.
> - Average time to claim payout: 0.5 years on HO; 4 years for WC.
> - Expected loss ratio: 50% for HO, 75% for WC
>
> Using a premium-based allocation, since the premium for each line is the same, we would allocate 50% of the surplus to each line.
>
> Using a reserve-based allocation, and assuming a steady state, we would require 100(75%)(4) = 300 in WC reserves, and 100(50%)(0.5) = 25 for HO, so we would allocate 12 times as much surplus to WC as to HO.

- Since a policy's risk depends on the size (and timing) of losses, still another method to allocate surplus would be to base on unpaid losses (in addition to, or in place of, premium writings).
- The policy form is also something worth considering when determining what allocation of surplus is most "appropriate." In particular:
    - Occurrence contracts pay on losses occurring during the policy period. Since claims-made contracts pay only on losses reported during the policy term, they are substantially less at risk of adverse deviation. For claims-made policies, there is no pure IBNR (incurred but not reported) after the policy expires, just IBNER (incurred but not enough reported).
    - Service contracts do not incur insurance risk, only expense risk.

- Retrospective rating eliminates insurance risk in the reimbursement layer (though not credit risk), though there is still insurance risk in the excess layers.

These and other variations in policy forms add a level of difficulty in allocating surplus, so should be considered in addition to premium and reserves.

Critiques of the IRR Method

IRR and Net Present Value (NPV) essentially measure the same thing – the IRR model solves for a discount rate that gives an NPV of zero. Even though they are mathematically equivalent for "accept or reject" decisions, an NPV analysis may be preferable for a variety of reasons.

1. **Ambiguous Results:** When the signs of the cash flows change more than once, IRR can yield multiple real roots. This is not a really big issue in insurance equity IRR analyses, since there is necessarily one sign change, from negative (investment of capital when policies are written) to positive releases of surplus as losses are paid, assuming IRR analyses not entirely dissimilar from the simplified model presented in the text. On that note…

2. **Oversimplifications:** Actual cash flows may show multiple sign changes, as a firm may need to raise additional capital mid-term due to adverse reserve development. An IRR model may be written to ignore this possibility, which is good for calculations, but makes no allowance for different events.

3. **Mutually Exclusive Projects:** Since the output of an NPV model is a dollar figure, while the output of IRR is a percent, if a firm were faced with two mutually exclusive projects, and decided based on IRR alone, they could potentially reject a project that would give them more cash, in favor of a project with a higher projected IRR. (Although realistically, it would be fairly foolish of the firm's management to base their decision on any one measure alone.)

4. **Reinvestment Rates:** A major drawback of the IRR model is that it implicitly assumes that a project with a higher IRR can be reinvested at that higher IRR (since this is what is used as the discount factor in the model). If a firm's cost of capital is 15%, a model producing an IRR of 20% would indicate a favorable project, but assumes the firm can attain a 20% rate of return by reinvesting funds.

   This argument however is of little consequence in insurance models, since an insurer can in fact earn the IRR by writing more policies that generate that IRR. Additionally, if the firm selects a premium rate so that IRR is equal to the cost of capital, then the issue is moot.

5. **Presentation Problem:** Another "problem" with IRR is that it may result in a misleading conclusion by a lazy interpreter. The IRR rule dictates that a venture is good if the IRR is greater than the cost of equity capital. But suppose that the IRR is positive – a regulator may view that as sufficiently high to justify the project. Going further, if the IRR is greater than the investment return (but less than the cost of capital), a regulator may not be willing to view the project as unprofitable. A presentation of NPV would make it clear that a project is unprofitable (negative dollars), while IRR does not.

Robbin (The Underwriting Profit Provision & IRR, ROE, and PVI/PVE)

The Robbin UW paper, written in 1992, presents several methods for determining a profit provision, including internal rate of return (IRR), return on equity (ROE), and ratio of present value of income to present value of equity (PVI/PVE). Those three methods from that paper are covered in the newer paper from 2007. Since the later paper is just an update of the three methods presented in the 1992 version, I'm combining them into one summary, and indicate the source where necessary, in case you want to refer back. Enjoy!

Background

This (pair of) paper(s) discusses different methods of determining the provision to be used for underwriting profit when determining premiums.

Historical Background (UW Paper)

Traditionally, a 5% underwriting profit provision was used as standard profit, except in Workers' Compensation lines, where a 2.5% load was used. Why, you may ask? For no reason, other than tradition. Starting around the 1970s-1980s, companies began considering more rigorous means of determining underwriting profit. In the face of rising competition, many insurance companies engaged in rate wars to the point of insolvency.

Because insurance in the United States is regulated by state rather than nationally, there are a variety of methods used to compute profit provision, some of which are discussed in the paper.

Underwriting Profit vs. Total Profit (UW Paper)

Note that underwriting profit is not the same as an insurance company's profit. Total profit includes underwriting profit, as well as investment profit. Thus, an insurance contract could be profitable, even when writing the contract at a loss. There are some other issues with determining profit that complicate things:

- **Measurement basis**: Ratemaking is done on a prospective policy-year basis, while return on equity is usually measured on a historical calendar year basis (GAAP ROE measures GAAP income as a ratio to GAAP equity). It can be difficult to reconcile the two, and even if we could define policy-year return and equity, we still need to adjust for the time value of money and select a target return.
- **Investment Income Offset:** Rates can be set to explicitly offset for investment income on policyholder-supplied funds (PHSF = premiums less loss and expense payments and declared profits). Investment income offsets from PHSF can be viewed on a prospective policy-year or a calendar year basis.

In the latter case, one must consider what portion of total investment income is generated specifically from PHSF – the remainder of the investment income is returned to the stockholders, so arguably should not be considered in the profit provision with regard to the policyholders.

The split between PHSF and stockholder-supplied funds can help gauge adequacy of total returns. Companies can have positive total profits even with negative underwriting profits, but positive total profits do not necessarily mean that a stock company has earned a fair return for stockholders.

Five Types of Underwriting Profit (UW Paper)

1. **Underwriting profit provisions** are used in manual rates and rate filings. *This type of profit is the focus of the paper.*
2. **Corporate target underwriting profit provisions** are used by management to determine sufficient premiums to meet stockholder expectations.
3. **Breakeven underwriting profit provisions** are those such that a stockholder could earn the same as a risk-free rate of return.
4. **Charged profit provision** is the rate obtained after considering experience, schedule rating, and other adjustments to the manual rate.
5. **Actual underwriting profits** are calculated *ex post* and can be used to evaluate the appropriateness of estimates.

Regulatory Issues (UW Paper)

Theoretically, under a perfect "rate of return regulation" approach, insurance companies would be restricted from earning excessive profits by simply examining their rate of return, similar to what is done in utility company regulation. The difficulty in this approach stems from the fact that measuring return is complicated due to the need to assign surplus to different lines of business.

Another theory of regulation contends that competitive markets should force the price of insurance to optimal levels. This seems the more appropriate theory since, unlike in the case of utility regulation, insurance companies do not have a monopoly in the industry.

Measures of Return (IRR Paper)

Return on an insurance policy can be looked at in three ways:

1. We can look at the return to an equity investor.
2. We look at the corporate return and apply to a policy. The GAAP Return on Equity (ROE) measure is often used to measure corporate calendar year (CY) return.
3. We can also extend the CY ROE measure to the life of the policy to better understand the overall profitability of a multi-year venture.

Overview of Profit Measures to be Presented (Both Papers)

Robbin presents various options to consider when setting a profit measure, summarized in the table below, but he does not dictate which is the "correct" option. Indeed, there is no "correct" option; the choice of provision method depends on several factors, including:

- External factors like regulatory requirements and market competitiveness
- Whether and how surplus should be reflected in the model

  The IRR paper gets into a little bit of detail about the issues with surplus. This is a concept that should be super familiar from many other places in the syllabus. Surplus could be set based on expected losses (commonly done in academic examples), which means that actual premium has no effect on needed surplus.

  Another complication is considering how surplus should evolve over the life of a policy. The paper uses surplus as a fixed percentage of the present value of unpaid losses – in real life, the percentage should vary with risk (by policy and development age).
- Whether and how risk should be reflected in the model
- Whether cash or income flows are used
- Discounting of cash flows
- Selection of target return
- Taxes

Profit Provision Model	Description
(FROM THE UW PAPER)	
Calendar Year Investment Offset	Reduce ("offset") profit load by an amount to account for calendar year investment income
Present Value Offset	Offset profit load by the difference in present value of reviewed versus reference line of business
Present Value Return on Cash Flow	Set profit provision such that PV(Income) = PV(Change in Equity) • Here, income includes underwriting and investment income, net of taxes. • Income is discounted at investment rate; equity is discounted at target rate of return.
Risk-Adjusted Discounted Cash Flow Model	Determine premium so that PV(Inflows) = PV(Outflows) • Inflows = premium • Outflows include losses, expenses, and income taxes. • Except for losses, we use a risk-free discount rate. Losses are discounted at the lower risk-adjusted rate, since they are uncertain.

Profit Provision Model	Description
(FROM THE IRR PAPER)	
Internal Rate of Return on Equity Flows	This is the standard IRR on equity flow method that looks for return that would be achieved by an equity investor. Profit is found by adjusting premium until yield on equity flows meets target.
Present Value of Income over Present Value of Equity (PVI/PVE)	Profit adjusted until present value of return (ratio of accounting income valued as of the end of year 1 to annualized equity) meets a target return value.
Growth Model Calendar Year Return Method	Set profit to achieve a selected target return, based on CY investment income. It becomes a growth model when we include an assumption for multi-year periods.

Before giving more information about each of the profit measures, the paper reviews some basic formulas for premium, profit, and combined ratio determinations. These might be familiar to you already.

$$\text{Premium} = \frac{\text{Losses} + \text{ALAE} + \text{Fixed Expenses}}{1 - \text{Variable Expense Ratio} - \text{Profit Rate}}$$

$$\text{Combined Ratio} = \text{Variable Expense Ratio} + \frac{\text{Losses} + \text{ALAE} + \text{Fixed Expenses}}{\text{Premium}}$$

$$\text{Underwriting Profit} = 1 - \text{Combined Ratio}$$

METHODS PRESENTED IN THE UW PAPER[1]

CY Offset, PV Offset, PV Cash Flow, Discounted Cash Flow

Calendar Year Investment Income Offset (UW Paper)

Description

The CY Investment Income Offset method offsets a traditional profit provision (U_0) to remove the effect of **calendar year** investment income, adjusted for income taxes and income from stockholder equity.

Process

1. Determine the ratio of after-tax investment income to invested assets, i_{AT}. This ratio depends on tax rates by class (e.g., bonds are taxed differently than are dividends) but the details of this determination is beyond the scope of this paper. The actuary may or may not choose to include realized and/or unrealized capital gains.

   The result of this ratio is denoted the *after-tax portfolio yield*.

2. Estimate policyholder-supplied funds (PHSF). We want to account for that fact that premiums generate only a portion of invested assets. The equation presented in the text to determine this is:

$$PHSF = \left[\frac{\text{UPR, net of Prepaid Acq. Exp} \\ -\text{Premiums Receivable}}{\text{Earned Premium}}\right] + \left[\text{Permissible Loss Ratio} \times \frac{\text{Reserves}}{\text{Incurred Loss}}\right]$$

   Note that each component of the formula is a ratio.

   - The UPR (unearned premium reserve) is stated as a portion of earned premiums, as are premiums receivable.
   - Prepaid expenses (such as commissions, premium taxes, other acquisition expenses, overhead) are netted out of the UPR since they have already been used, and do not generate investment income.
   - Likewise, premiums receivable (as a portion of earned premiums) are subtracted out; they also cannot be invested.
   - The second bracketed portion determines the portion of reserves that are investible.
   - The ratio of reserves to loss is calculated on a calendar-year basis, and the actuary may use the average of multiple years to add some stability to this component. Multiplying by the permissible loss ratio gives us a ratio with respect to premiums, as in the first bracketed portion.

[1] You can find the exhibits for this paper in the directory: "Other Files > **Excel Supplements > Exhibits_Robbin_UWProfit.xlsx**"

Why don't we instead directly just use an actual ratio of reserves-to-premiums, you ask? Well, that's a good question. The answer is that doing so would potentially distort results in cases of rapid growth or decline, or changes in premium adequacy. In the method presented here, we can keep the ratio consistent with that used in the prospective rate.

3. Determine the final underwriting profit provision as: $\boldsymbol{U = U_0 - i_{AT}PHSF}$

   (Since the profit provision is calculated from the permissible loss ratio, and vice versa, the calculations should be done iteratively.)

Advantages/Disadvantages

A key advantage here is that calendar year data are available from Annual Statements, so pulling data is pretty easy. Also (except in the case of rapid growth or decline or volatile reserve adequacy), calendar year investment income is pretty stable. One way to adjust for rapid changes in volume would be to adjust the loss reserve portion of investment income to scale for differences in loss ratio. Another disadvantage is the lack of general economic theory to support the calculation.

Example

Given the following, use the calendar year investment income offset method to determine a provision for underwriting profit.

- Traditional underwriting profit provision: 5%
- After-Tax investment yield: 6%
- Earned Premiums: 1000
- Unearned Premiums: 100
- Premiums Receivable: 20
- Pre-Paid Acquisition Expense: 20%
- Loss ratio used in prospective rates: 75%
- Historical ratio of reserves to incurred loss: 30%

This section of the syllabus really does not lend itself naturally to graphic arts. Please enjoy this picture of a robin.

We need to use: $U = U_0 - i_{AT}PHSF$. We are given the traditional profit provision and the after-tax investment yield, so we have: $U = 5\% - (6\%)PHSF$

PHSF are the funds available for investing, which come from premium reserves and loss reserves.

Investible Premium Reserves

UPR, net of prepaid expenses: $100(1 - 20\%) = 80$

Less premiums not yet received: $80 - 20 = 60$

So, investible assets from premiums, as a ratio of earned premium, are: $60 \div 1000 = 0.06$

Investible Loss Reserves

The historical portion of losses in reserves is 30%. We convert that to a ratio with respect to premium by multiplying by the permissible loss ratio. (Our units become $\frac{\text{reserves}}{\text{loss}} \times \frac{\text{loss}}{\text{premium}} = \frac{\text{reserves}}{\text{premium}}$). So, we have $30\% \times 75\% = 0.225$

Total Investible Funds: $0.06 + 0.225 = 0.285$

Throwing these back into the original formula, we have an underwriting profit provision of:

$$U = 5\% - (6\%)PHSF$$
$$U = 5\% - 6\%(0.285) = \mathbf{3.29}\%$$

Present Value Offset Method (UW Paper)

Description

The PV Offset method also starts with a traditional underwriting profit provision, U_0, to arrive at a final provision for underwriting profit. In this case, the offset is the difference between the present value of losses for the reference line versus that under review. This adjustment is intended to reflect the difference in investment income potential between lines of business.

To determine the present value of each line, you need an assumed loss payment pattern, and of course a discount rate. Determining a discount rate can be a tricky dicky. Some options are the estimated new money yield, since this seems in line with prospective ratemaking. Or, we could use portfolio yield from a recent year, which is the current yield. This is less in line with prospective ratemaking, but preferred in this case for its stability and verifiability.

We could also account for taxes in this method by using an after-tax interest rate for discounting. This technically accounts only for taxes on the present value of loss, not all taxes, but it's beyond the scope of the exam to delve further.

Nonetheless, the impact of taxes should be considered. There are two basic options:

- Taxes can be calculated prospectively by determining an average of prospective tax rates, weighted by type of investment asset.
- Alternately taxes can be calculated retrospectively, using actual historical income tax rates, assuming that future tax rates will be similar.

Process

1. Select an appropriate discount rate to determine the present value of the reviewed and referenced lines of business.
2. Offset the traditional profit provision by the difference in the two figures:

$$U = U_0 - PLR\left(PV(x_{\text{reference}}) - PV(x_{\text{reviewed}})\right)$$

Here, $U_0 = \text{traditional profit provision}$; $PLR = \text{permissible loss ratio}$, and the present values are based on the cash flows of the respective lines, at the selected discount rate. The cash flows, x, refer to the **portion** of losses that are paid out, so that the undiscounted sum of x_i is 1.

Advantages/Disadvantages

One difficulty of this method is, as in the calendar year investment income method, determining an appropriate interest rate, in this case to be used for discounting. However, this method is not distorted by rapid growth or decline.

Example

A firm is calculating a profit provision for a new line of business. The new line of business pays out 100 over four years, with payments in each year of: 20, 20, 30, 30.

The reference line of business uses a profit provision of 5% and pays out 100 over two years, in two equal payments.

Use the present value offset method to determine an appropriate profit provision, assuming the firm has selected an interest rate of 6% to be used for discounting. Assume an (undiscounted) loss ratio of 100%.

The present value of the reference line is: $50v + 50v^2 = 91.67$, where $v = \frac{1}{1.06}$

Similarly, the present value of the reviewed line is: $20v + 20v^2 + 30v^3 + 30v^4 = 85.62$

The difference in these values, relative to the total payment is: $\frac{91.67-85.62}{100} = 0.0605$.

Thus, the losses in the reference line are worth 0.0605 more than the line under review and we need to offset the traditional profit margin by that amount.

So, the indicated profit provision is $0.05 - 0.0605 = -\mathbf{1.05}\%$.

Present Value Cash Flow Return Model (UW Paper)

Description

In the present value cash flow return model (PVCF), as the name sort of implies, the present value of cash flows (at an investment rate of return r_i) is set equal to the present value of change in equity (at the target rate of return r_r).

Cash flows used in this model include underwriting cash flows (from premiums, losses, and expenses) and investment income (on the portion of equity that can be invested), and income taxes. This model does not directly adjust for investment income on policyholder-supplied funds; that income is already implicitly considered in underwriting cash flows.

Process

Determine premium required such that: $\text{PV}(\Delta\text{Equity}; r_r) = \text{PV}(\text{CF}; r_i)$

Again, note that cash flows include both underwriting and investment income cash flows.

Advantages/ Disadvantages

This method requires an allocation of surplus, and estimates regarding future equity flows. Also conceptually this method is difficult to reconcile since it does not really measure any one sort of profit. It's not GAAP return on equity, since it does not use GAAP income, and it would be difficult to coerce into GAAP accounting since the timing of underwriting cash flows is different than that of GAAP income. However, the method is attractive because it does directly measure underwriting cash flows, which is intuitive with profit measurements.

Example: (From the text)

Consider a policy requiring 100 in equity for the first year, and 40 for the second, and 0 thereafter. (Thus, the equity changes are 100, -60, -40).

If the firm targets a 20% return, the present value of the change in equity at the end of the first year (see note below), is: $1.2 \times 100 - 60 - 40 \div 1.2 = 26.67$.

The indicated profit provision would be that which would allow the present value of expected cash flows to equal 26.67.

Note: A student pointed out that the discounting in this method was inconsistent between the example in the paper and Exhibit 6. The example evaluates PV of profit as of end-of-year (EOY), while Exhibit 6 discounts to beginning of year (BOY).

I asked Dr. Robbin about the intended treatment. His response:

"The exhibit was made to track with Mahler's method, but the text is trying to make the PV return comparable to the PVI of PVI/PVE method. PVI should be evaluated at the end of year one. The Mahler Total PV Return method evaluates all PV as the start of the first year. That is what is done in the exhibit. I got confused when writing the text and incorrectly applied an end of first year valuation."

So, BOY discounting is the intended treatment; note though that this is based on an off-syllabus conversation, so you should not reasonably expect your exam grader to know that.

When in doubt, good answer documentation always helps clarify things like that. "I used BOY discounting consistent with the exhibit/ I used EOY discounting consistent with the example."

Restating the above example using BOY discounting would indicate a profit provision such that PVCF is: $100 - 60 \div 1.2 - 40 \div 1.2^2 = \mathbf{22.22}$.

Risk-Adjusted Discounted Cash Flow Model (UW Paper)

Description

The risk-adjusted discounted cash flow (RADCF) model aims to balance the present value of premium with that of losses and expenses, including income taxes. The "risk-adjusted" portion of the DCF comes from the fact that losses, being less certain than the other components, are adjusted at the risk-adjusted rate, which is typically lower than the risk-free rate that is used to discount the other parts.

Process

Determine the premium such that the present value of premium is equal to the present value of losses, expenses, and income taxes. In Section 9 of the IRR paper, note that taxes are broken into the following components:

- Taxes on Investment Income on Surplus and Premium net of Expense
- Taxes on Underwriting Income from Premium less Expense
- Taxes (negative) on losses

Losses (and associated taxes) are discounted at a risk-adjusted rate r_A, while the other components are discounted at the risk-free rate r_f:

$$\mathbf{PV(Premium; r_f) = PV(Expenses + Taxes; r_f) + PV(Losses; r_A)}$$

The selection of the risk-adjusted rate is an important parameter in this method. One approach for selecting the rate would to select one that compensates the insurer for the insurance risk. An alternative approach uses a risk-adjusted rate based off the CAPM: $r_A = r_f + \beta(R_M)$. Recall that β captures the systematic risk of an investment, and R_M measures the *excess* of market return over the risk-free rate $(R_M = r_M - r_f)$.

The paper also calculates present value as of **the end of the first year** (not to time 0). The goal of this measure is to balance out cash flows at the end of the first year – the indicated profit provision is then backed out based on the undiscounted cash.

Note also that the taxes discounted at the risk-free rate are the taxes on the investment income from surplus (surplus only; the paper does not make an allowance for investment income from reserves) and the non-loss portion of underwriting cash. Taxes on losses are determined as a portion of the present value of losses, so are implicitly calculated at the risk-adjusted rate.

Advantages/ Disadvantages

A key advantage here is simply the intuitive appeal. Also, allocation of surplus is not a critical assumption here (though it has the drawback of not allowing us to directly reflect STAT requirements). However, the method is sensitive to selection of discount rate, which in turn is sensitive to the determination of beta. You've seen in BKM that this is no easy process with stocks, but is even more difficult when dealing with individual lines of business. Finally, there is no easy way in this method to reflect reserve discounting since it does not affect the underwriting cash flows used in this model.

Example 1

Determine the appropriate premium for a firm, using the risk-adjusted discounted cash flow model, and the following assumptions:

- Premium collected at policy inception
- Expenses paid on inception = 30
- Taxes paid out on investment income on surplus at the end of the first year = 10
- No taxes on underwriting income
- Losses paid out at the end of each of the first 3 years, in the pattern: 30, 25, 20
- Risk-free rate = 5%
- Selected beta: -0.8
- Market return rate = 10%

In the RADCF model, expenses and taxes on non-loss components are discounted at the risk-free rate. So the expense component of premium, discounted to the end of the first year is: $v = \frac{1}{1.05} \rightarrow 30/v + 10 = 41.50$

To determine the present value of losses, we need the risk-adjusted rate, which comes from the CAPM: $r_A = r_f + \beta \cdot R_M = 5\% - 0.8(10\% - 5\%) = 0.01$.

Thus, the loss component of premium is: $w = \frac{1}{1.01} \rightarrow 30 + 25w + 20w^2 = 74.36$

So, the premium collected (as of the end of the first year) should be 41.50 + 74.36 = **115.86**.

Example 2 (based on Exhibit 7)

Given the following information:

	Present Value (at t = 1)	Average Discount Factor
Premium	$106.84	1.05
Loss	$62.58	n/a
Expense	$42.25	n/a
FIT on Inv Income	$0.95	1.03

The average discount factor shown in the table is the average discount applied weighted by cash flows.

- Fixed Expenses = $15
- Federal Income Tax (FIT) rate for underwriting cash = 34%
- Variable Expenses = 25%
- Undiscounted Loss = $65

1. Confirm that the values in the table conform with the risk-adjusted discounted cash flow method:

   In RADCF, $\text{PV}(\text{Premium}; r_f) = \text{PV}(\text{Expenses} + \text{Taxes}; r_f) + \text{PV}(\text{Losses}; r_A)$. The only component missing from the table is taxes on underwriting cash. We determine that as $34\%(106.84 - 62.58 - 42.25) = \$0.68$

   Premium is (roughly) equal to the sum of the discounted cash flows $(62.58 + 42.25 + 0.68 + 0.95 = 106.46)$.

   Note that in the present values displayed, losses are discounted at a risk-adjusted rate, while the other components are discounted at the risk-free rate.

2. Determine the indicated underwriting profit provision.

$$\text{Combined Ratio} = \text{Variable Expense Ratio} + \frac{\text{Losses} + \text{ALAE} + \text{Fixed Expenses}}{\text{Premium}}$$

$$\text{Underwriting Profit} = 1 - \text{Combined Ratio}$$

   We know from exhibit:

   - Variable Expense Ratio = 25%
   - Losses (assume ALAE included) = $65
   - Fixed Expenses = $15

   We can determine premium by rolling back to t = 0: $\$106.84 \div 1.05 = \$101.78$

Thus, the underwriting profit = $1 - \left[25\% + \frac{\$65 + \$15}{\$101.78}\right] = \mathbf{-3.60\%}$

METHODS PRESENTED IN THE IRR PAPER

IRR, PVI/PVE, CY ROE[2]

Return Measures Background (IRR Paper)

Before presenting the other three methods, from the newer paper, Robbin presents some additional background regarding their uses.

We use return measures to determine appropriate indicated prices for policies. The prices should reflect risk, as well as management's risk-return preferences. In light of this, Robbin uses theoretical surplus allocations instead of actual, to ensure consistency with risk appetites. (It is beyond the scope of the paper to determine how to allocate surplus.) For purposes of the return models, the risk considered is concentrated on loss timing and size.

The models presented in this paper (IRR, PVI/PVE, CY ROE) all consider the use of surplus (equity), in contrast to other discounted cash flow models, which consider the flow of (underwriting) cash, thereby disregarding the impact from the accounting treatment of expenses and surplus.

Basic Definitions

The paper presents some basic formulas for determining underwriting income and equity. Underwriting income in a period j is simply earned premium, less incurred losses and expenses $\left(U_j = EP_j - IL_j - IX_j\right)$. GAAP and STAT recognize initial expenses differently – in STAT, expenses are incurred following a fixed pattern, while in GAAP, expenses are incurred as premium is earned.

The difference in the treatment of initial expenses is called the Deferred Acquisition Cost (DAC**).** Assuming for simplicity that the only difference between GAAP and STAT is in recognition of expenses, the required GAAP equity Q_j, in terms of the required statutory surplus S_j for a policy lasting n years is given by:

$$Q_j = \begin{cases} S_0 + \text{DAC} & j = 0 \\ S_j & 0 < j < n \\ 0 & j = n \end{cases}$$

Investible assets are defined as the sum of statutory reserves (unearned premium reserves, loss reserves, and expense reserves) and statutory surplus. (This is identical to the definition used in the Feldblum paper.) Invested assets are the difference between investable assets and amounts recoverable.

IRR on Equity Flows Method (IRR Paper)

Description

In this method we determine the IRR demanded based on a given set of equity flows. Recall that equity flows are the flows of money between an investor and the company.

[2] You can find the exhibits for this paper in the directory: "Other Files > **Excel Supplements > Exhibits_Robbin_IRR.xlsx**"

Process

We determine equity flows in each year. Equity flow in a given year is income, less increase in equity.

- For the zeroth year (policy inception), there is no income (assuming premiums and expenses are not yet recognized), so the equity flow is simply $-Q_0$, where the required equity is determined by company management.

  This flow is always negative because of the initial requirement of capital from investors, as well as the equity associated with the DAC (equity in the Unearned Premium Reserve), which requires expenses to be incurred upfront rather than recognized as premium is earned.

- For years 1 through n, equity flow is the difference between income and change in required equity.

Advantages/ Disadvantages

- A commonly noted disadvantage of IRR model is the potential for multiple roots. This would occur in the case where there are multiple sign changes. This could happen if we used IRR on underwriting cash flows, but in the IRR Equity Flow model, this is generally not an issue. After the initial negative equity requirement, equity flows are all negative and then all positive, or else all positive.

  Releasing the surplus will always create a positive equity flow, though initial flows may be negative depending on how and when reserves are considered. Therefore, in the IRR Equity Flow model, there is no issue with multiple solutions.

- The second issue with the IRR model is the implicit assumption that funds can be reinvested at the IRR rate, which is not necessarily the market rate.

Example (based on Exhibit 6 of the paper):

Given the following assumptions for a certain insurance policy, determine the IRR.

Year	Income	Equity
0	0	38.2
1	5	15.7
2	3.5	5.3
3	1.2	0
4	0	0

We calculate the equity flows as income less change in required equity. So I've added two columns to the table. Change in Equity is the difference in current equity and prior equity, and equity flow is the income column less the change in equity column.

Year	Income	Equity	Change in Equity	Equity Flow
0	0	38.2	38.2	-38.2
1	5	15.7	-22.5	27.5
2	3.5	5.3	-10.4	13.9
3	1.2	0	-5.3	6.5
4	0	0	0	0

The IRR is solved with the following:

$$0 = -38.2 + \frac{27.5}{1 + IRR} + \frac{13.9}{(1 + IRR)^2} + \frac{6.5}{(1 + IRR)^3}$$

This gives us **IRR = 16.00%**

Note that this development of IRR is slightly different than the method as presented in the *simplified* example from the Feldblum paper. If you'd like to reconcile them, then the Feldblum method would need to be expanded so that:

- Taxes are paid as they are incurred.
- Underwriting cash flows are based on earned and incurred cash, rather than paid cash (this is not a difference in treatment – the simple example does not differentiate between paid and incurred, though the expanded example in the Feldblum appendix does).
- In the Feldblum paper, equity flow = cash – assets (reflects the additional equity needed to be supplied from the shareholder, where assets include reserves and surplus). In this paper, equity flow is closer to Cash – ΔAssets, where the assets are only related to surplus requirements (essentially assuming that shareholders fund surplus only, not the reserves)

It is also important to note though that this method is pretty much the same as shown in the expanded Feldblum example.

PVI/PVE Method (IRR Paper)

Description

There are no surprises in the description of this method. The ratio of present value of income to present value of equity calculates just that, the ratio of PVI to PVE. One thing to be careful with though is that "present value" of income is stated as of the end of the first year. This is intuitively appealing since that is in line with how ROE is calculated.

Process

Determine the present value of income (as of the end of the year), and the present value of equity, and then take the ratio.

$$\frac{PVI}{PVE} = (1+r_I)\sum_{j=1}^{n}\frac{I_j}{(1+r_I)^j} \Big/ \sum_{j=0}^{n-1}\frac{Q_j}{(1+r_Q)^j}$$

r_I is used to denote the rate used to discount income; r_Q is used to denote the rate used to discount equity.

While different discount rates can be used to discount income and equity, the authors prefer to use the same rate, ideally the cost of capital, since the company can borrow at this rate.

Advantages/ Disadvantages

- Using the cost of capital as a discount rate represents an improvement over the IRR method's issue of the implicit assumption of reinvesting at the IRR.

- This method is fairly straightforward using annual cash flows but if we wish to determine on some other basis, it's a little trickier. For example, if we would like to determine on a quarterly basis, it's easy enough to pull quarterly income amounts for discounting, and their sum will roughly be the annual amount. However, the same is not true for equity – the denominator will be about four times as large as before. To account for the discrepancy, we can either view the return as a quarterly effective return, or annualize the return by dividing by the sum of the quarterly discount factors, which will be around 4.

Example (based on Exhibit 6 of the paper, same as IRR example above)

Given the following assumptions for a certain insurance policy, determine the PVI/PVE. Assume that a 16% cost of capital is deemed an appropriate discount rate for both income and equity.

Year	Income	Equity
0	0	38.2
1	5	15.7
2	3.5	5.3
3	1.2	0
4	0	0

Letting $v = 1/1.16$:

PV(Income) = $5v + 3.5v^2 + 1.2v^3 = 7.68$. Note that this is as of time 0, so to bring to end of year 1, we have: $7.68 \times 1.16 = 8.91$

PV(Equity) = $38.2 + 15.7v + 5.3v^2 = 55.67$

So, PVI/PVE = $8.91 \div 55.67 = \mathbf{16.00}\%$

Obviously right now you are amazed by the fact that this works out to be the same value as what was calculated for IRR above. This demonstrates an important result of this method:

When the rate used to discount income and equity are the same as the IRR, then PVI/PVE = IRR.

Under this interpretation, the IRR method is just a specific case of PVI/PVE when the discount rate is equal to the IRR (which of course you would need to find first).

Growth CY ROE Method (IRR Paper)

Description

In this model, we go beyond the single policy model and instead make a more realistic assumption that the firm continues to write business beyond the first year, and desires to calculate profit for the ongoing venture. Suppose then that equity grows at some constant rate, g. If in the single policy model, we have cash flow in a given year of CF_1, then the policies written next year we would have cash flow of $(1+g)CF_1$. The year after that, we'd have $(1+g)^2 CF_1$, and then $(1+g)^3 CF_1, (1+g)^4 CF_1$, and so forth.

Example

Suppose a policy generates income at the end of each year for three years, according to the pattern: 10, 5, 2. If the company writes 1000 policies in year 1, and continues to write the same book of policies each year for five years, but increases its volume by 10% per annum, the cash flow until all of the policies expire will be:

The company will write 1000, 1100, 1210, 1331, and 1464 policies in each of the first five years.

Each policy will generate 10, 5, and 2 of income its 1st, 2nd, and 3rd years. Thus we have:

<table>
<tr><th></th><th>Policy 1</th><th>Policy 2</th><th>Policy 3</th><th>Policy 4</th><th>Policy 5</th></tr>
<tr><td>CY 1</td><td>1000×10
$= 10{,}000$</td><td colspan="4">Not yet written</td></tr>
<tr><td>CY 2</td><td>1000×5
$= 5{,}000$</td><td>1100×10
$= 11{,}000$</td><td colspan="3">Not yet written</td></tr>
<tr><td>CY 3</td><td>1000×2
$= 2{,}000$</td><td>1100×5
$= 5{,}500$</td><td>1210×10
$= 12{,}100$</td><td colspan="2">Not yet written</td></tr>
<tr><td>CY 4</td><td rowspan="4">Expired</td><td>2,200</td><td>6,050</td><td>13,310</td><td>Not yet written</td></tr>
<tr><td>CY 5</td><td rowspan="3">Expired</td><td>2,420</td><td>6,655</td><td>14,640</td></tr>
<tr><td>CY 6</td><td rowspan="2">Expired</td><td>2,662</td><td>7,320</td></tr>
<tr><td>CY 7</td><td>Expired</td><td>2,928</td></tr>
</table>

Income for each of the first through seventh calendar years is therefore:

Calendar Year	Income
1	10,000
2	5,000 + 11,000 = 16,000
3	2,000 + 5,500 + 12,100 = 19,600
4	21,560
5	23,715
6	9,982
7	2,928

Process

Notice that the growth rate from year 3 to year 4 is $\frac{21{,}560}{19{,}600} - 1 = 10\%$. The growth rate from year 4 to year 5 is $\frac{23{,}715}{21{,}560} - 1 = 10\%$.

When the growth rate of cash flows equals the growth rate of each policy, we say the business is in the **Equilibrium Growth Phase**. If we specifically consider end of year j income I_j and beginning of period equity Q_j as our cash flows, then, for any year in the equilibrium growth phase, we have:

$$ROE = \frac{\sum_{j=1}^{n} I_j(1+g)^{-(j-1)}}{\sum_{j=1}^{n} Q_j(1+g)^{-j}}$$

We can pull out a common factor from the numerator to get this:

$$ROE = \frac{(1+g)\sum_{j=1}^{n} I_j(1+g)^{-j}}{\sum_{j=1}^{n} Q_j(1+g)^{-j}}$$

Since equity is using beginning of year flows and income is using end of year flows, let's instead bring them to a common base of end of year flows. So instead of indexing equity to start at the beginning of year 1 and end in year n, let's have it start at the end of year 0 and end in year $n - 1$. We get:

$$ROE = \frac{(1+g)\sum_{j=1}^{n} I_j(1+g)^{-j}}{\sum_{j=0}^{n-1} Q_j(1+g)^{-j}}$$

At this point you should note that, if you let $g = r_I = r_E$, this formula is exactly the same as that used for PVI/PVE. So we can add to our important result from earlier.

When the rate used to discount income and equity are the same as the IRR, then PVI/PVE = IRR.

Also, if the rate used for discounting is the same as the constant growth rate, then PVI/PVE = CY Growth ROE = IRR.

So, the CY Growth ROE method provides another way to interpret IRR. If the growth rate is IRR, then ROE is also IRR, so all income supports growth – the IRR is the maximal self-sustaining rate.

Advantages/ Disadvantages

The main advantage here is that ROE is a tried-and-true measure, and everyone loves it.

Example (based on Exhibit 6 of the paper):

Suppose we are given the following flows of equity and income for a growing business, in its equilibrium growth phase. Further suppose that the business is growing at a constant rate – let's say… 16%. Determine the growth model return on equity.

Year	Income	Equity
0	0	38.2
1	5	15.7
2	3.5	5.3
3	1.2	0
4	0	0

It's **16%.** The work will look identical to that shown for the PVI/PVE method.

If instead we assume a growth rate of 10%, we would have:

Letting $v = 1/1.10$:

Numerator = $(1.1)(5v + 3.5v^2 + 1.2v^3) = 9.17$

Denominator = $38.2 + 15.7v + 5.3v^2 = 56.85$

Return on equity = $9.17/56.85 = 16.13\%$, which is not the IRR from before.

(The point is that $g = ROE \leftrightarrow g = IRR$.)

General Notes (IRR Paper)

Intuitively, a higher surplus requirement will indicate a higher profit provision, though not necessarily following a direct inverse relationship. Similarly, it is clear that an increase in interest rates (without increasing target returns) will reduce the profit provision, since additional investment income can be used to offset underwriting losses. Finally, increasing the duration of loss payouts will decrease the indicated profit provision.

SUMMARY

The "offset" methods determine underwriting profit directly:

Calendar Year Investment Offset

- $U = U_0 - i_{AT}PHSF$; $PHSF = \begin{bmatrix}\text{UPR, net of Prepaid Acq. Exp} \\ -\text{Premiums Receivable}\end{bmatrix} + \left[\text{Permissible Loss Ratio} \times \frac{\text{Reserves}}{\text{Incurred Loss}}\right]$
    - Formulas above are as ratios to Earned Premium
- No underlying economic theory supporting this calculation process
- Can pull CY data easily; fairly stable unless rapid growth or decline
- Must determine after-tax investment rate
- Relies on selection of traditional profit load

Present Value Offset

- $U = U_0 - PLR\big(PV(x_{\text{reference}}) - PV(x_{\text{reviewed}})\big)$
- Not distorted by rapid growth or decline
- No need to select target return or allocate surplus
- May be difficult to determine discount rate
- Relies on selection of appropriate reference line of business and traditional profit load

The other methods give ways of determining adequate premium subject to some measure; then a profit provision can be backed out.

Present Value Return on Cash Flow

- Determine premium such that $\text{PV}(\Delta\text{Equity}; r_r) = \text{PV}(\text{CF}; r_i)$, then back out underwriting provision
- Must allocate surplus and assume future equity flows

Risk-Adjusted Discounted Cash Flow Model

- Determine premium such that $\text{PV}(\text{Premium}; \text{r}_\text{f}) = \text{PV}(\text{Expenses} + \text{Taxes}; \text{r}_\text{f}) + \text{PV}(\text{Losses}; \text{r}_\text{A})$
    - Present value is determined as of $t = 1$
- Surplus allocation not critical, only flows through to investment income, but this has drawback of limiting the ability to reflect STAT requirements
- Need to determine a risk-adjusted rate (validity of CAPM-beta selection process is arguably even more questionable than the process of surplus allocation)
- Does not naturally reflect reserve discounting

Internal Rate of Return on Equity Flows

- Determine premium so that IRR of equity flows satisfies target
- Won't have multiple roots when using equity flows
- Method assumes that funds can be reinvested at the IRR

Present Value of Income over Present Value of Equity (PVI/PVE)

$$\frac{PVI}{PVE} = (1 + r_I) \sum_{j=1}^{n} \frac{I_j}{(1 + r_I)^j} \Bigg/ \sum_{j=0}^{n-1} \frac{Q_j}{(1 + r_Q)^j}$$

- Can explicitly use cost of capital for discounting, so better than IRR
- May be a pain to calculate non-annual cash flows

Growth Model Calendar Year Return Method

$$ROE = (1 + g) \sum_{j=1}^{n} I_j (1 + g)^{-j} \Bigg/ \sum_{j=0}^{n-1} Q_j (1 + g)^{-j}$$

- Can be distorted by growth
- Familiar metric to management
- Requires allocation of surplus (equity)

Kreps (Riskiness Leverage Models)

The Kreps paper presents a general formulation of risk load for total cash flows. His risk load produces additive co-measures at any level of detail. He uses a riskiness leverage ratio to capture the idea that some total outcomes are riskier per dollar than are others. The **riskiness leverage function** itself is arbitrary.

Returning to the idea of the importance of allocating capital, Kreps notes that the allocation is used for creating a pricing risk load that is really allocating the cost of capital. The goal is to allow for a solid statistical basis for the allocation of this arbitrary amount of required capital. Total capital (including surplus) will be the sum of the expected losses plus the risk load.

Desirable Qualities of an allocatable risk load formulation

1. Allocatable to any level of definition
2. Risk load allocated for any sum of random variable should be the sum of the risk load amounts allocated individually
3. Same formula used to calculate risk load for any subgroup or group of groups

Ideally, the formulation will be additive and allocatable, so that senior management can allocate to regions, regional managers can allocate to line of business, and the individual line of business allocations will add back up to the original.

The riskiness leverage function itself may vary, and will reflect management's risk tolerance, but Kreps' framework is designed to work for any choice of risk function. Once chosen, this enables management to quantitatively make management decisions, such as choosing an appropriate reinsurance program.

The Formulation

The total amount of capital C required to support a set of liabilities with mean μ and risk load R is given by $C = \mu + R$. A riskiness leverage model would have form $R = \int f(x)\,(x - \mu)L(x)dx$. This form indicates that riskiness leverage **depends only on the sum of the individual variables, and it reflects that all dollars are not equally risky (especially those that trigger regulatory flags).** The total risk load R will be the sum of the risk factors for each individual risk, R_k, and likewise total capital C will be the sum of the capital allocated to each individual risk, C_k. Now all we need to find is an adequate riskiness leverage model, $L(x)$. Leverage is calculated on a firm-wide basis.

We note that some variables may have negative risk loads, when they take on values less than their mean. This is not undesirable, as hedges and reinsurance would exhibit this behavior (net negative loss at the tails).

Form Development

Section 3 of the paper presents a formulation for using covariance as leverage and presents some basic calculus to demonstrate that the covariance choice has the desired property of being an additive allocation measure (the sum of covariances between individual variables and the total is the covariance of the total itself). It leads to another formulation of the risk load for a loss variable x_k, given as:

$$R_k = \int \overline{dF}(x_k - \mu_k)L(x)$$

where $\overline{dF}$ is used to abbreviate $f(x_1, x_2, \dots, x_n)dx_1dx_2 \dots dx_n$.

An integral of this form cannot be evaluated analytically except for very simple cases. Kreps uses Excel to demonstrate numerical calculations of more advanced cases.

Coherent Risk Measures – Background

This is not described in the paper, but you should probably know what the paper is referring to when it describes the lack of coherence of the risk load.

A risk measure, ρ, is **COHERENT** if it satisfies four properties for all losses x and y in any risk:

Property	Symbols	Insurance Meaning
Subadditivity	$\rho(x+y) \leq \rho(x) + \rho(y)$	Covering two losses is cheaper than covering each of those two losses alone (diversification)
Positive homogeneity	$\rho(\lambda x) = \lambda\rho(x), \lambda \in \mathbb{R}$	Changing a loss by some factor requires the same factor to be applied to assets required (e.g., in quota share)
Monotonicity	$\rho(x) \leq \rho(y)$ when $x \leq y$	Assets required either increase or don't decrease as loss increases
Translational invariance	$\rho(x+\alpha) = \rho(x) + \alpha$	If loss increases by some amount α, so do the total assets needed

The properties of this risk load are:

- $R(c) = 0$. Risk load is 0 for constant losses. This makes sense – If loss is known, no additional capital should be needed outside of the value of the loss.
- $R(\lambda X) = \lambda R(X)$. Risk load scales with currency change, provided the leverage ratio is homogeneous of order zero, or $L(\lambda X) = L(X)$. This will be the case if L is defined to be a ratio of currencies (e.g., $x/\mu, x/\sigma, x/S$, where we take a ratio with respect to mean, standard deviation, or surplus). The use of surplus would be beneficial for several reasons – it's guaranteed available and easily liquefiable capital. Also, the amount of surplus a company holds is directly related to its risk of ruin.

However, the risk load is not necessarily *coherent*, as the choice of L will dictate whether the subadditivity property is satisfied.

Examples of Riskiness Leverage Functions

Recall that risk load is formed as: $R = \int f(x)\,(x-\mu)L(x)dx$. The following are examples of leverage ratios $L(x)$ that could be chosen to determine risk load.

Function / Form[3]	Characteristics
Risk-neutral $L(x) = c$	• $c =$ some constant • Risk Load is 0 • Appropriate for risk of ruin when ruin is remote, or when measuring risk of not meeting plan when that is not an important consideration
Variance $L(x) = \frac{\beta}{S}(x-\mu)$	• $\beta =$ some constant; S=company surplus • Considers entire distribution • Considers upside and downside risks to be equally undesirable • Quadratic relation between risk load and loss • Total capital becomes $C = \mu + S, S = \sqrt{\beta \text{Var}[X]}$ • We could use some other form like β/μ, which would imply riskiness leverage does not depend on surplus available (unless the level of surplus were embedded in the choice of β.
TV@R $L(x) = \frac{\theta(x - x_q)}{1-q}$	• q is the desired percentile, θ is what Kreps calls the step or index function, which is 1 if the argument is positive, and 0 if negative. ***Note:*** *Kreps is silent as to value for $\theta(0)$; the step function does not have a consistent definition of 0; common choices are 0, ½, or 1.* • Results in riskiness leverage that is 0 up to a point, constant thereafter. The constant is chosen to replicate TV@R. • Coherent (TV@R is a coherent measure by design). • Allocated capital is average value of situations in excess of chosen probability level. • Considers only the right tail of the distribution.

[3] I don't believe the expression of forms to be particularly useful for exam questions as there is little value in them being tested (although that fact has not historically precluded questions from appearing), so just presenting them to show how they relate to each other.

Function / Form[3]	Characteristics
V@R $L(x) = \frac{\delta(x - x_q)}{f(x_q)}$	• $\delta(x)$ is the Dirac delta function[4] • Riskiness leverage concentrated at one point, the V@R level • Not coherent (V@R is not sub additive) • Does not consider the shape of the loss distribution, except to determine the relevant value x_q
Semi-Variance $L(x) = \frac{\beta}{S}(x - \mu)\theta(x - \mu)$	• Considers only downside of variance • Results in a risk load that is non-zero only when results are adverse • Quadratic relation • Used for risk of not achieving plan, when ruin is not being considered
Mean Downside Deviation $L(x) = \beta \cdot \frac{\theta(x - \mu)}{1 - F(\mu)}$	• Results in riskiness leverage ratio of zero below the mean, and constant above it • Natural naïve measure; assigns capital for bad outcomes in proportion to their severity • Can be used to measure risk of not achieving plan, when ruin is not being considered • Multiplier β can be chosen based on probability of severely weakening surplus • This is just excess tail-value-at-risk at $x_q = \mu$
Proportional Excess $L(x) = \frac{h(x)\theta[x - (\mu + \Delta)]}{x - \mu}$	• Individual allocation for any outcome is pro-rata on contribution to excess over mean • Here Δ is some constant and either $\Delta > 0$ or $h(\mu) = 0$.

Properties of the Riskiness Leverage Ratio

The riskiness leverage framework helps to answer the question about how much surplus is needed to sustain the business, given its risks. Now we consider the question of how to choose an appropriate leverage function.

[4] Dirac delta function is a generalized function that is zero at all values except 0, and has an integral of 1 over the real numbers. Can consider it as a function with value 0 for all reals except at $x = 0$, where it is a line of infinitely high length.

For management, desirable qualities of riskiness leverage ratios are:

- Downside measure
- Constant for excess that is small, compared to capital
- Become larger for excess significantly impacting capital
- Go to zero (or not increase) for excess significantly exceeding capital – "once you are buried, it doesn't matter how much dirt is on top."

For regulators, the desirable qualities are more "streamlined":

- Zero until capital is seriously impacted
- Not decrease (because would impact state guaranty fund)

For a regulator, TV@R may be an appropriate risk measure, where the quantile is chosen to match some specified fraction of surplus, such as a function of the below form, where α represents some specified fraction of surplus:

$$L(\boldsymbol{x}) = \frac{\theta(x - \alpha S)}{1 - F(\alpha S)}$$

This would still leave business risk, since for values larger than the percentile specified, the company would become insolvent.

Excel Simulation

Kreps provides an Excel simulation of losses to demonstrate allocation ratios between two lines A and B using different risk measures. In this company, management is not concerned with just risk of ruin, but ongoing risk as well. It uses TV@R to formulate its risk appetite as:

"For the x percent of possibilities of net income that are less than $(income corresponding to $x\%$), we want the surplus to be a prudent multiple of the average value so that we can go on in business."

The results show that the portions allocated to each line are not particularly sensitive to the TV@R level chosen for any percentile smaller than 10 – in each case, Line B needs approximately 6 times as much surplus as does A. The amount of surplus required can change depending on what portion of the business is written in Line B rather than Line A (if the company increases the volume assigned to Line A, the required surplus will decrease).

If writing less volume of line B were not an option for some reason (e.g., regulatory restrictions or indivisible policies), management could instead consider purchasing reinsurance to provide surplus relief.

Mango (An Application of Game Theory: Property Catastrophe Risk Load)

This paper looks to explore methods of determining appropriate risk loads for property catastrophe insurance.[5] A couple of common methods use Marginal Surplus and Marginal Variance, which determine the impact on risk load when a new portfolio is added to an existing book of business. The big drawback of these methods is that they exhibit **order dependency** – that is, the marginal impact of Portfolio B added to Portfolio A, is not generally the same as the impact if B were added first. These issues are not unique to marginal surplus and variance – the same issue is present with value-at-risk.

Used in this paper and tested not infrequently on the exam are formulas from way back in prelim days. For a refresher, the paper assumes a binomial probability of occurrence, so, for a catastrophe model output with modelled loss events denoted L_i, with occurrence probability p_i, we have:

$$\mathrm{E}[L] = \sum_i L_i \times p_i$$

$$\mathrm{Var}[L] = \sum_i L_i^2 \times p_i \times (1 - p_i)$$

The covariance between two portfolios called L and M would be given as:

$$\mathrm{Cov}[L, M] = \sum_i L_i \times M_i \times p_i \times (1 - p_i)$$

The variance of the combined portfolio is: $\mathrm{Var}[L + M] = \mathrm{Var}[L] + \mathrm{Var}[M] + 2\mathrm{Cov}[L, M]$

[5] You can find the exhibits for this paper in the directory: “Other Files > **Excel Supplements > Exhibits_Mango.xlsx**”

Example 1:

Suppose we have two portfolios X and Y. A catastrophe model output produces the following potential scenarios and indicated probabilities, sorted by descending order of total loss:

Event	Probability	$Loss_X$	$Loss_Y$	$Loss_{X+Y}$
1	2%	1000	100	1,100
2	1%	800	20	820
3	3%	600	150	750

We can determine expected values, variance, and covariance using the above formulas.

Item	Result	Calculations
$\mathbf{E}[\boldsymbol{X}]$	46	$(2\%)(1000) + (1\%)(800) + (3\%)(600)$
$\mathbf{E}[\boldsymbol{Y}]$	6.7	$(2\%)(100) + (1\%)(20) + (3\%)(150)$
$\mathbf{Var}[\boldsymbol{X}]$	36,412	$(2\%)(98\%)(1000)^2 + (1\%)(99\%)(800)^2 + (3\%)(97\%)(600)^2$
$\mathbf{Var}[\boldsymbol{Y}]$	854.71	Similar
$\mathbf{Var}[\boldsymbol{X}+\boldsymbol{Y}]$	46,742	Similar
$\mathbf{Cov}[\boldsymbol{X},\boldsymbol{Y}]$	4,737.4	$(2\%)(98\%)(1000)(100) + (1\%)(99\%)(800)(20) + (3\%)(97\%)(600)(150)$

- We can double check the covariance by using the variance:

$$\mathrm{Var}[X+Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\mathrm{Cov}[X,Y]$$
$$46{,}742 = 36{,}412 + 854.71 + 2\mathrm{Cov}[X,Y]$$

$$\mathrm{Cov}[X,Y] = \mathbf{4,737.6}$$

Marginal Surplus and Marginal Variance Methods

Suppose we set a required surplus, $\boldsymbol{V = zS - R}$**,** where z is determined from the normal table at some probability, S is the standard deviation of loss, and R is the expected return.

If we decompose R into R_0 (return before new account) and R_1 (return after new account is added), and likewise for standard deviation, we have:

$$V_0 = zS_0 - R_0$$
$$V_1 = zS_1 - R_1$$

The additional surplus needed is therefore: $V_1 - V_0 = z(S_1 - S_0) - (R_1 - R_0)$. This additional surplus is the **marginal surplus**. Denoting the additional risk load required, $R_1 - R_0$, as r and incorporating required return on surplus (based on management goals, market forces, and risk appetite), y, gives us a marginal risk load of:

$$r = \left[\frac{yz}{1+y}\right][S_1 - S_0]$$

Derivation of above:

Before the addition of the new account, the return on surplus required is: $yV_0 = y(zS_0 - R_0)$

After the addition of the new account, required return on surplus is: $y(zS_1 - R_1)$

The marginal risk load needed is:

$$r = y(V_1 - V_0) = y\big(z(S_1 - S_0) - (R_1 - R_0)\big)$$

By definition, $r = R_1 - R_0$, so this becomes:

$$r = y(z(S_1 - S_0) - r)$$

$$r = yz(S_1 - S_0) - yr$$

$$r + yr = yz(S_1 - S_0)$$

$$r(1 + y) = yz(S_1 - S_0)$$

$$r = \frac{yz}{1 + y}(S_1 - S_0) \blacksquare$$

The **marginal variance** method can be derived similarly, and indicates a risk load of $r = \lambda[\text{Var}(M) + 2\text{Cov}(L, M)]$ when a new account *M* is added to existing portfolio *L*. In this case, $\lambda = \left[\frac{yz}{1+y}\right] \div S_1$

Example 2: From the previous example, we have:

- $\text{Var}[X] = 36{,}412 \rightarrow \sigma_X = 190.82$
- $\text{Var}[Y] = 854.71 \rightarrow \sigma_Y = 29.24$
- $\text{Var}[X + Y] = 46{,}742 \rightarrow \sigma_{X+Y} = 216.20$

If we desire to use the **marginal surplus** method to determine the additional risk load_required when portfolio Y is added to portfolio X, assuming we use a z-value of 2.0 (corresponding to a loss exceedance probability of about 2.275%), and a required return on surplus of 20%, we have:

Original Risk load (using only portfolio X): $R_X = \left[\frac{0.2(2.0)}{1+0.2}\right][190.82] = \mathbf{63.61}$

Additional Risk load after adding Portfolio Y: $r_Y = \left[\frac{0.2(2.0)}{1+0.2}\right][216.20 - 190.82] = \mathbf{8.46}$

The total risk load would be $63.61 + 8.46 = \mathbf{72.07}$

The risk load on the combined account is the same: $R_{X+Y} = \left[\frac{0.2(2.0)}{1+0.2}\right][216.20] = \mathbf{72.07}$

Note here that if portfolio Y had been first, the risk load for Y alone would have been $R_Y = \left[\frac{0.2(2.0)}{1+0.2}\right][29.24] = \mathbf{9.75}$). This is <u>greater than</u> the above indication, a property mathematicians denote as "uncool."

We could instead use the **marginal variance** method to determine the additional surplus required to add portfolio Y to existing portfolio X:

Original Risk load: $R_X = \left[\frac{0.2(2.0)}{1+0.2}\right] \div 216.20 \cdot [36{,}412] = \mathbf{56.14}$

Additional Risk load for adding Y: $r_Y = \left[\frac{0.2(2.0)}{1+0.2}\right] \div 216.20 \cdot [46{,}742 - 36{,}412] = \mathbf{15.93}$

Note that if Portfolio Y had been first, our answer here would be lower than the above indication, which is also uncool.

$$R_Y = \left[\frac{0.2(2.0)}{1+0.2}\right] \div 216.20 \cdot [854.71] = \mathbf{1.32}$$

The total risk load would be $56.14 + 15.93 = \mathbf{72.07}$, as above.

Mathematical Uncoolness 1 – Order Dependence

Example 2 demonstrates an unfortunate property of the marginal methods – **order dependence**. Looking at the risk load for Portfolio Y, we do not get the same result when adding Portfolio Y to X as we'd get if we'd added Portfolio X to Y.

Example 3: Suppose in example 2, instead of adding Y to X, we added X to Y. The additional risk load for taking on X, using the marginal surplus method, would be:

$$r = \left[\frac{0.2(2.0)}{1+0.2}\right][216.20 - 29.24] = 62.32$$

We note that this is 1.29 lower than what was originally calculated when X was a stand-alone portfolio. This is mathematically uncool.

Using the marginal variance method, the additional risk load would be:

$r = \left[\frac{0.2(2.0)}{1+0.2}\right] \div 216.20 \cdot [46{,}742 - 854.71] = 70.75$, which is 14.61 more than originally calculated for X in the marginal variance method where X was added first. This too, is mathematically uncool.

To summarize, note that the marginal surplus method is **subadditive** – the risk load for X, when it is added on to Y, is less than the risk load when X is alone. Also, the sum of the marginal risk loads (risk load for X when added to Y plus the risk load for Y when added to X) is less than the actual risk load required. The opposite case is true for marginal variance – it is **super-additive**.

	Risk Load For …				
Method	**X alone**	**X, if added to Y**	**Y, if added to X**	**Sum**	**Actual X + Y**
Marginal Surplus	63.61	62.32	8.46	70.78	72.07
Marginal Variance	56.14	70.75	15.93	86.68	72.07

Mathematical Uncoolness 2 – Sub- and Super-Additivity

Example 3 demonstrates more unfortunate properties of the marginal methods:

- The marginal surplus method gives a sub-additive result – that is, $MS_X + MS_Y \leq MS_{X+Y}$. This results because surplus uses the square root function, which is also sub-additive (square root of a sum is less than the sum of square roots).

An insurance example of a sub-additive function is the insurance premium for a group of insureds – the total insurance premium for a group would be less than sum of the insurance premiums each individual member would pay if they were written by themselves.

- The marginal variance method gives a super-additive result – $MV_X + MV_Y \geq MV_{X+Y}$. This is because the sum of the marginal variances double counts the covariance between each of the portfolio pieces.

Note: A student pointed out that the inequalities here are confusing because they are the opposite mathematical definition of sub-additivity and super-additivity. In general, for a function ρ:

- **Sub-additive**: $\rho(X+Y) \leq \rho(X) + \rho(Y)$
- **Super-additive**: $\rho(X+Y) \geq \rho(X) + \rho(Y)$

The inequalities as shown in the definitions in the section above are correct as stated, even though they appear reversed according to the definition.

Proof:

Marginal variance, super-additive

$\text{Var}[X+Y] = \text{Var}[X] + \text{Var}[Y] + 2\text{Cov}(X,Y)$

Assuming positive covariance: $\text{Var}[X+Y] > \text{Var}[X] + \text{Var}[Y]$, as in definition of super-additivity.

By definition, $\text{MV}_X = \text{Var}[X+Y] - \text{Var}[Y] = \text{Var}[X] + 2\text{Cov}(X,Y)$

Likewise: $\text{MV}_Y = \text{Var}[Y] + 2\text{Cov}(X,Y)$

Thus, $\text{MV}_X + \text{MV}_Y = \text{Var}[X+Y] + 4\text{Cov}(X,Y) \rightarrow \text{MV}_X + \text{MV}_Y \geq \text{MV}_{X+Y}$

So, the signs are opposite of what is expected, but do correctly follow from the definition of super-additivity.

Marginal surplus, sub-additive

$$\sigma_{X+Y} = +\sqrt{\sigma_X^2 + \sigma_Y^2 + 2Cov(X,Y)} = +\sqrt{\sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y}$$

Since $(\sigma_X + \sigma_Y)^2 = \sigma_X^2 + \sigma_Y^2 + 2\sigma_X\sigma_Y$, it follows that $\sigma_X^2 + \sigma_Y^2 + 2\rho_{XY}\sigma_X\sigma_Y \leq (\sigma_X + \sigma_Y)^2$, since $\rho_{XY} \leq 1$.

Therefore, $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$, definition of sub-additivity.

Then $\text{MS}_X = \sigma_{X+Y} - \sigma_Y$; $\text{MS}_Y = \sigma_{X+Y} - \sigma_X$.

Thus, $\text{MS}_X + \text{MS}_Y = 2\sigma_{X+Y} - (\sigma_X + \sigma_Y) \leq \sigma_{X+Y}$ since $\sigma_{X+Y} \leq \sigma_X + \sigma_Y$.

Since $\text{MS}_{X+Y} = \sigma_{X+Y}$, we have shown $\text{MS}_X + \text{MS}_Y \leq \text{MS}_{X+Y}$. Again, opposite of what is expected, but follows from definition.

Mathematical Coolness: Renewal Additivity

A cool property for risk loads to have would be **renewal additivity**. Unlike the marginal surplus method, which would tend to undervalue the risk associated with renewing an account, or the marginal variance method, which would go too far in the opposite direction, a renewal additive method would be just right.

Background to Game Theory

The paper presents some basic information about Game Theory. Game theory looks to model interactions between two players in a "game," any situation involving set rules and outcomes. One type of game is the cooperative game, in which players work together to achieve a common goal. Cooperative games are often based in the concept of transferrable utilities. A utility is transferable if it can be transferred to another player, without loss.

Cooperative games with transferrable utilities are characterized by:

- Players with some shared cost or resource
- Ability to negotiate/ bargain to share resources/ costs among players
- Conflicting objectives – each player wants to maximize his own outcome

Mathematically, we can describe such a game where the total amount to be allocated is given by a **characteristic function**, associating a real number $v(S)$ to each coalition S of players.

A sub-additive characteristic function is one such that, for two disjoint coalitions S and T, $v(S \cup T) < v(s) + v(T)$. A super-additive function satisfies $v(S \cup T) > v(S) + v(T)$.[6]

When considering methods of allocation, there are certain desirable standards.

- Allocations must be additive (sum should equal total amount to be allocated).
- Coalitions should be satisfy both **individual and collective rationality**. These are coalitions that are said to be stable[7], wherein individuals are not worse off for having joined a coalition; collectively, the coalition should be such that no subgroups would be better on their own.

[6] The text uses strict inequalities; technically the inequalities should be $\leq, \geq$ as in the regular definitions of sub- and super-additivity.

[7] The text says that stability roughly translates to "fair." This is not completely accurate; outcomes may be stable, but not fair. As a simple example, suppose there is an investment that costs $1,000, and has a guaranteed payoff of $2,000. If two friends have only $500 each, they can propose to invest, and split the profits 50-50. This is stable and fair – neither is worse off for having joined the coalition. It is also fair since they have even investments and even payoffs. Another outcome would be to split the profits in any other way, say 70-30. This is still a stable outcome – both are better off in the coalition than out of it. It is clearly unfair though. Outcomes may be stable without being fair.

The set of additive allocations that satisfy the fair and stable conditions constitute the **core** of the game.

Tying these ideas back into property catastrophe loads, consider a game in which a risk load is assigned to a portfolio of accounts. In this game, the accounts are the players, the portfolio represents a coalition, and we can use the portfolio's variance or standard deviation as a characteristic function. The goals of this game are:

- Each player would like to minimize their allocation of risk load
- Allocation of risk should be fair and objective
- Allocation should be additive

With respect to the third bullet, neither the marginal surplus nor the marginal variance methods work. The next two methods presented will satisfy the additivity property.

Renewal Additive Method 1: Shapley Value

The **Shapley Value** assigns a specific distribution of a total surplus in a non-arbitrary method and is elegant in its simplicity and ability to achieve renewal additivity. For two portfolios L and M, the Shapley Value (as applied to marginal variance) for adding portfolio M to portfolio L is given by:

$$\text{Shapley Value} = \text{Var}(M) + \text{Cov}(L, M)$$

This is the same as the marginal variance formula EXCEPT it does not double-count the covariance component. The beauty here is that the Shapley Value satisfies renewal additivity, regardless of the order in which the portfolios are added. (Note: The Shapley Value could also be used for the marginal standard deviation method, but because of the square roots and such, the work is messy, so the authors chose not to present it.)

Example 4: Continuing with the example from earlier, where we had:

- $\text{Var}[X] = 36{,}412$
- $\text{Var}[Y] = 854.71$
- $\text{Var}[X + Y] = 46{,}742, \ \sigma_{X+Y} = 216.20$
- $\text{Cov}[X + Y] = 4{,}737$

$$(\text{Shapley Value})_X = 36{,}412 + 4{,}737 = 41{,}149$$

$$(\text{Shapley Value})_Y = 854.71 + 4{,}737 = 5{,}592.1$$

Letting z = 2.0 and y = 20% again would give us indicated risk loads of:

$$R_X = \frac{2(0.2)}{1.2 \times 216.20}(41{,}149) = 63.44$$

$$R_Y = \frac{2(0.2)}{1.2 \times 216.20}(5{,}592.1) = 8.62$$

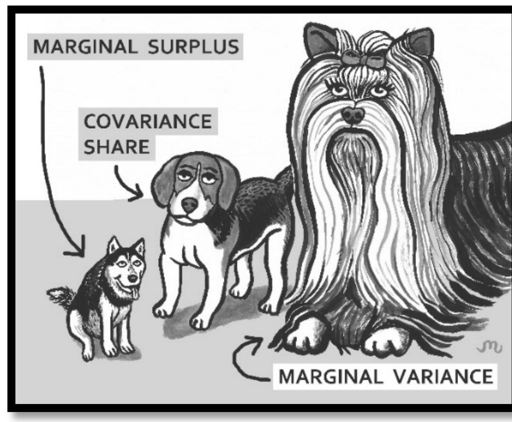
The sum is the same as the risk load required for the two loads. amazing

Adding to our table from earlier, we see how cool the Shapley Value method is.

	Risk Load For …				
Method	**X alone**	**X, if added to Y**	**Y, if added to X**	**Sum**	**Actual X + Y**
Marginal Surplus	63.61	62.32	8.46	70.78	72.07
Marginal Variance	56.14	70.75	15.93	86.69	72.07
Shapley Method	56.14	63.44	8.62	**72.06**	**72.07**

But wait, there's more…

Renewal Additivity Method 2: Covariance Share Method



Putting aside our awe at the greatness of the Shapley Method for now, we recognize that while the method necessarily works because it apportions the covariance in equal parts, there is no real reason for doing it that way. Even though X and Y do not necessarily contribute equally to the covariance, they still share equally in this approach.

We could simply generalize the Shapley Method so that instead of sharing 50%, 50%, we choose some other allocation of percents that seems more equitable.

The method in the text weights the covariance to each portfolio for each event by the proportionate amount of loss it contributes to that event. In the example that we've been using, this would allocate significantly less capital to Event Y (since it is relatively smaller than X). This is a desirable quality, and this method will still maintain the additivity that we want.

Example 5:

Recalling our events from earlier, and adding to our table:

Event	Probability	$Loss_X$	$Loss_Y$	$Loss_{X+Y}$	Cov(X,Y)	CS_X	CS_Y
1	2%	1000	100	1,100	1,960	3,564	356.4
2	1%	800	20	820	158.4	309.1	7.73
3	3%	600	150	750	2,619	4,190	1,048
				Sum	4,737	8,063	1,412

Sample Calculations

Covariance:

- Event 1: $(2\%)(98\%)(1000)(100) = 1{,}960$
- Event 2: $(1\%)(99\%)(800)(20) = 158.4$
- Event 3: $(3\%)(97\%)(600)(150) = 2{,}619$

Weights:

- Event 1: $w_X = 1000 \div 1100$
- Event 2: $w_X = 800 \div 820$
- Event 3: $w_Y = 150 \div 750$

Covariance Share:

- Event 1: $CS_X = 1000 \div 1100 \times 1960 \times 2$
- Event 2: $CS_X = 800 \div 820 \times 158.4 \times 2$
- Event 3: $CS_Y = 150 \div 750 \times 2619 \times 2$

Notice that the total covariance is 4,737, which is also **half** the sum of the covariance shares. This is what we desire, since in the total variance formula, the covariance is added in twice.

The Covariance Share Method would give the following risk loads:

$$(\text{Covariance Share})_X = 36{,}412 + 8{,}063 = 44{,}475$$

$$(\text{Covariance Share})_Y = 854.71 + 1{,}412 = 2{,}267$$

$$R_X = \frac{2(0.2)}{1.2 \times 216.20}(44{,}475) = 68.57; \; R_Y = \frac{2(0.2)}{1.2 \times 216.20}(2{,}267) = 3.50$$

Our final entry in our table also satisfies the desired properties, much like the Shapley Method, but in a more equitable way:

	Risk Load For …				
Method	**X alone**	**X, if added to Y**	**Y, if added to X**	**Sum**	**Actual X + Y**
Marginal Surplus	63.61	62.32	8.46	70.78	72.07
Marginal Variance	56.14	70.75	15.93	86.69	72.07
Shapley Method	56.14	63.44	8.62	**72.06**	**72.07**
Covariance Share	56.14	68.57	3.50	**72.07**	**72.07**

Wow.

OBJECTIVE D REVIEW QUESTIONS

*Questions marked with a **(☆)** contain data in the **Excel Supplement for Exercises**.*

*Some questions also contain solutions in Excel. They are marked with a **(★★)**.*

Ferrari (The Relationship of Underwriting, Investment, Leverage, and Exposure to Total Return on Owners' Equity)

Fer-1. Explain why P/S and I/A tend to move in opposite directions.

Fer-2. Explain why P/S and U/P tend to move in the same direction.

Fer-3. Explain why U/P and I/A tend to move in the same direction.

Fer-4. Given the following, determine the firm's return on equity.

- Investment return on assets: 7%
- Reserves + liabilities: 3,000
- Stockholders' equity: 2,000
- Underwriting profit: 300
- Premium Income: 4,000

McClenahan (Insurance Profitability)

McC-1. How does the purchase of an insurance policy resemble an opportunity cost for the policyholder? Why is the risk-free rate of return the appropriate discounting rate used to measure the value of this cost?

McC-2. What are the shortcomings with using equity as a denominator for rate of return? How does using a denominator of sales address those shortcomings?

McC-3. (Based on Tables 1 and 2 from the reading) Given the following assumptions for a private passenger automobile policy:

- Premium: 100,000, paid on inception
- Loss Ratio: 0.65
- Expense Ratio: 0.35, with all expenses paid in the middle of the first year
- Loss Payout Pattern (paid at middle of Years 1 – 5, respectively): 0.25, 0.35, 0.20, 0.12, 0.08
- Risk-free rate of return: 6%
- All underwriting cash flows are invested.

Determine the indicated profit, in dollars, and as a percent of premium. What does this profit represent?

McC-4. Four companies, A, B, C, and D each have the same expected loss distribution, with mean \$300. Companies A and B propose a rate of \$315, while C and D propose a rate of \$330. Companies A and C have a premium-to-surplus ratio of 3:1; Companies B and D have a ratio of 1.5:1.

Describe numerically how using the concept of rate equity versus return on equity would lead to troublesome results.

Feldblum (IRR)

IRR-1. List several risks against which surplus is intended to protect. Which occur only during the policy period?

IRR-2. The IRR method may be used with equity to help price policies, but there are some potential drawbacks to the method. List and describe five issues with the IRR method.

IRR-3. **(★★)** Given the following assumptions for a one-year policy, determine the IRR of equity flows.

- Premium collected at the beginning of the year
- Expenses are 20% of premium, paid at inception
- Losses are targeted at an 80% loss ratio, paid out equally at the end of the next two years
- Surplus is held to satisfy a reserves:surplus ratio of 2:1
- Surplus is determined based on unpaid losses
- Surplus and reserves are invested at a 7% return.
- Ignore taxes.

IRR-4. **(★★)** Given the following assumptions for a one-year policy, determine the IRR of equity flows.

- Premium of 100 collected at the beginning of the year
- Expenses of 10, incurred and paid on inception
- Loss paid out at the end of years 1, 2, 3 in the pattern: 40, 40, 30
- Surplus is held to satisfy a reserves:surplus ratio of 3:1
- Surplus is determined based on unpaid losses and expenses
- Investible assets (reserves and surplus) earn a 10% return
- Shareholders are used to fund both reserves and surplus.
- Ignore taxes.

IRR-5. Repeat the previous problem, except assume that surplus was allocated using an alternate method, which indicated that the ratio should be 2:1. If the cost of capital is 8%, what decision would be indicated by the IRR in each problem?

IRR-6. Repeat the previous problem (2:1 reserves to surplus), except assume that surplus is committed at policy initiation and released after all losses are paid.

IRR-7. Your firm writes three lines of business, A, B, and C. The annual written premium, expected loss ratio, and average time between claim occurrence and payment for each line are summarized below.

Line	Written Premium	Expected Loss Ratio	Average payout lag
A	$1,000	0.80	0.5 years
B	$1,000	0.70	1 year
C	$1,000	0.60	5 years

Given $1,000 in surplus, how much is allocated to each line of business:

a. Using a premium-based allocation?
b. Using a reserve-based allocation, and assuming steady state?

IRR-8. Your firm writes two lines of business. The annual written premium, expected loss ratio, and average time between claim occurrence and payment for each line are summarized below.

Line	Written Premium	Expected Loss Ratio	Average payout lag
A	$2,000	0.80	0.5 years
B	$1,000	0.70	1 year

Determine the IRR on equity flows for Line A, if capital is allocated by written premium. Assume the following:

- One-year policy period
- Losses on average occur halfway through the policy period.
- Premiums are collected at the beginning of the policy period.
- Expenses of 20% of premium are paid at the beginning of the policy period.
- The total firm capital held is $1,500.
- Surplus and loss reserves are invested and earn a 7% return.
- Equity flows occur annually.
- Surplus is committed on inception and released after all losses are paid out.

IRR-9. A junior actuary at your firm is looking over the capital allocation by line of business. Intuitively, he believes that the surplus should be allocated based on the size of the business, so he does not understand why the surplus for each line is not roughly proportional the amount of business written. What are some other factors you can explain to him that should be considered in the surplus allocation?

IRR-10. Suppose that for a certain project, your company's cash flows for a series of years, starting at time 0 are: -45, 140, -55, -140, 100.

a. What would be the IRR on cash flows for the project? You may want to use a graphing application to solve your equation.[8]

b. What complication about the IRR method does this exemplify?

c. Why is this issue not generally relevant when determining IRR on equity flows?

[8] https://www.desmos.com/calculator

IRR-11. (★★) Given the following assumptions for a one-year policy:

- $5,000 in premium collected at the beginning of the year
- Expenses are $900, with half paid at inception, and half paid the next year.
- Expected loss ratio is 90%, with losses paid out equally over three years.
- Surplus is held to satisfy a reserves:surplus ratio of 3:2
- Surplus is determined based on unpaid losses and expenses
- Investible assets include reserves and surplus, and are invested at 5%
- The cost of capital is 6%.
- Ignore taxes.

a. What is the IRR on equity flows?

b. Based on the IRR results, your company asserts that the premiums need to be increased, but regulators disagree, also based on the IRR results. Explain what the issue may be, and what problem with the IRR method this exemplifies.

Robbin (The Underwriting Profit Provision & IRR, ROE, and PVI/PVE)

Ro-1. Explain why a company may choose to write a contract at a loss.

Ro-2. Explain the difficulties in reconciling profit targets with actual realized profit.

Ro-3. What are the five types of underwriting profit?

Ro-4. State three methods for determining return on an insurance policy.

Ro-5. Explain the difference between utility companies and insurance companies in terms of rate regulation.

Ro-6. A disadvantage of a profit measure that uses equity is in the difficulty of determining appropriate equity. What does the paper cite as one advantage a method using equity flows has over other methods that use underwriting cash?

Ro-7. Assuming they are using identical inputs, why might an actuary using the risk-adjusted discounted cash flow method to determine a profit provision come to a substantially different conclusion than an actuary determining a profit provision using the PVI/PVE method?

Ro-8. When determining an appropriate profit provision measure, what factors should the actuary consider?

Objective D Review Questions

This next section of exercises is quite honestly dreadfully boring[9], but important, as it is to help you get a good handle on the various methods presented.

Ro-9. With regard to the calendar year investment income offset method:

a. Write or describe the formula, defining each input.

b. What is the purpose of determining policyholder-supplied funds? How are they determined?

c. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-10. With regard to the present value offset method:

a. Write or describe the formula, defining each input.

b. What should be considered in determining a discount rate?

c. How can we account for taxes?

d. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-11. With regard to the present value cash flow return method:

a. Write or describe the formula, defining each input.

b. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-12. With regard to the risk-adjusted discounted cash flow return method:

a. Write or describe the formula, defining each input.

b. Why are losses discounted at a different rate than other cash flows?

c. How does Robbin set the risk-adjusted rate? What are the potential pitfalls of that specific method?

d. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-13. With regard to the IRR method:

a. Write or describe the formula.

b. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-14. With regard to the PVI/PVE method:

a. Write or describe the formula, explaining each input.

b. What does Robbin state as the advantages and/or disadvantages of this method?

[9] "ASM does not have a discussion of stimulation, but considering how boring the manual is, maybe it would be a good idea" – Abraham Weishaus, actuarialoutpost.com (RIP Outpost ✌️)

Ro-15. With regard to the Growth CY ROE method:

a. Write or describe the formula, explaining each input.

b. What does Robbin state as the advantages and/or disadvantages of this method?

Ro-16. Of the seven types of profit provision measures mentioned in the paper, list which one(s) apply to each of the following:

a. Requires use a traditional profit load
b. Specifies use of calendar year data
c. Requires allocation of surplus (equity)
d. Requires a pre-established discount rate to determine profit

This concludes the boring section of exercises. The next exercises are really fun!

Ro-17. Given the following, determine the policyholder-supplied funds.

- Written Premiums: 1,000,000
- Earned Premiums: 700,000
- Prepaid Acquisition Expense: 80,000
- Premiums Receivable: 40,000
- 3-year PLR: 60%
- CY Reserves: 750,000
- CY Incurred Loss: 700,000

Ro-18. Use the information below to determine the indicated profit provision using the CY Investment Income Offset Method.

- Earned Premium: 80,000
- Unearned Premium: 30,000
- Pre-paid expense ratio: 15%
- Premiums Receivable: 10,000
- Ratio of reserves to incurred loss: 1.0
- Permissible Loss Ratio: 70%
- Traditional U/W Profit Provision: 5%
- After-tax portfolio yield: 8.9%

Ro-19. Determine the after-tax portfolio yield assumed, using the below.

- Earned Premium: 100,000
- Unearned Premium: 70,000
- Pre-paid expense ratio: 20%
- Premiums Receivable: 20,000
- Ratio of reserves to incurred loss: 0.7
- Permissible Loss Ratio: 65%
- Traditional U/W Profit Provision: 5%
- CY Investment Income Offset method indicated profit provision: 4.8%

Ro-20. Determine the indicated profit provision, using the present value offset method assuming the following:

- Discount rate of 3.75% per annum
- PV(Reference Line Loss Paid Out, as a portion of total losses) = 0.865
- PV(Reviewed Line Losses Paid Out, as a portion of total losses) = 0.872
- Loss Ratio: 70%
- Traditional U/W Profit Provision: 5%

Ro-21. Determine the indicated profit provision, using the present value offset method assuming the following:

- Discount rate of 6.25% per annum
- Projected Loss Ratio: 65%
- Traditional U/W Profit Provision: 2.5%
- Loss payout pattern as indicated below (as a portion of total losses)

Time (Years)	**0.5**	**1**	**1.5**	**2**
Reference Line	30%	20%	40%	10%
Reviewed Line	45%	10%	45%	0%

Ro-22. Given the following (from Exhibit 6A of the reading):

	Present Value
Premium	$103.22
Loss	$57.34
Expense	$40.19
Investment Income on Surplus	$2.70
Changes in Equity	$5.54

Tax Rate = 34%

The present values used are in accordance with those appropriate for the present value cash flow return model.

a. What is the after-tax underwriting profit?

b. What is the after-tax investment income?

c. Given that the present values are appropriate for the PVCF method was used, what type of discount rate was used to determine each value?

Ro-23. Given the following assumptions for a one-year policy:

- Premium is paid at the beginning of the policy
- Premium-to-Surplus Ratio = 2:1
- Equity-to-Surplus Ratio = 1.3:1
- Investment yield = 7%
- Investment income is credited semi-annually
- Target rate of return = 15%
- Income Tax Rate = 35%
- Surplus equals investible equity
- Surplus is committed when the policy initiates, and released 1 year after policy expiry
- Payout pattern of losses and expenses:

Time	Loss	Expense
0	0	15% of premium
0.5	300	0
1	400	50
1.5	500	0
2	200	0
3	100	0

a. Determine the premium the firm should charge, and the profit provision, as indicated by the present value cash flow return model.

b. If the tax rate changes to 21%, determine the new charged premium and profit provision.

Ro-24. Given the following:

- 1,240 of premium received on January 1 of Year 1; determined based on the present value cash flow return model
- 200 in expenses paid on inception; 50 in expenses paid the following year
- 500 in claims paid on December 31 of Year 1, and 500 paid on December 31 of Year 2
- Management requires a 2:1 ratio of undiscounted reserves:surplus
- Surplus:equity ratio = 1:1
- Surplus equals investible equity
- Investment rate of return = 5%
- Ignore taxes.

Determine the target rate of return on equity.

Ro-25. A policy is priced at an underwriting profit of -3.60%, based on the risk-adjusted discounted cash flow (RADCF) model.

- Risk-free Rate = 8.00%
- Average Market Return = 10.50%
- Liability Beta = -0.750
- Income Tax Rate = 34%
- Undiscounted Federal Income Tax on Investment Income = 1
- Undiscounted Losses are twice as much as undiscounted expenses.
- The below payment patterns for losses and expenses apply.

Year	Premium	Loss	Expenses	FIT on Surplus
0	1	0.5	0.6	0
1	0	0.3	0.3	0.5
2	0	0.2	0.1	0.5

Determine the undiscounted policy premium.

Ro-26. Use the risk-adjusted discounted cash flow method to determine the underwriting profit provision for a one-year policy written by company XYZ, given the following assumptions:

- Half of premium collected on initiation of policy; second half of premium collected on expiry of policy.
- Expenses of 15% of premium are paid on initiation, and expenses of 500 are paid at the end of year 1.
- Losses are paid out fully by the end of 4 years, subject to the pattern 1000, 4000, 4000, 1000 at the end of each of the respective first four years.
- Surplus is ½ of undiscounted loss and expense reserves.
- Surplus equals investible equity.
- Expected return on investments = 6%
- Expected return on XYZ’s stock based on the CAPM = 15%
- XYZ’s CAPM Beta = 1.3
- Market premium = 7%
- Liability beta = -0.75
- Underwriting income tax rate = 21%
- Investment income tax rate = 15%

Ro-27. **(★★)** Given the following assumptions:

- Premium of $1,260 collected on inception and earned in even increments at the end of the first three years.
- Loss of $1,000 is paid out over the end of each of three years in the pattern: $500, $300, $200.
- Variable expenses of 16% paid on inception, and fixed expenses of 15 paid at the end of each of the next three years.
- Expenses and losses are incurred as they are paid.
- Premium-to-Surplus ratio of 3.0
- Surplus committed on inception and released when all losses have been paid.
- Investment yield = 7%
- Income tax rate on underwriting income = 21%
- Income tax rate on investment income = 25%
- The company holds reserves for unearned premium, expenses, and losses.
- Surplus and reserves are investible assets.

Determine the IRR on equity flows, using equity flows defined as in the Robbin Paper.

Ro-28. **(★★)** Given the following assumptions, determine the profit provision required to achieve an IRR of 14%.

- Premium is collected and fully earned on policy inception.
- Losses incurred immediately and paid out at the end of the first 3 years, in the pattern $900, $50, $300.
- Variable expenses of 20% paid and recognized on policy inception.
- Surplus is based on a 1.5:1 ratio of undiscounted reserves to surplus and is released as reserves are divested.
- The company holds reserves for unearned premium, expenses, and losses.
- Surplus and reserves are investible assets and have an expected return of 8%.
- Income tax rate = 21% (for investment income and underwriting income)

Ro-29. **(★★)** (From Exhibit 2, Sheet 1 of the IRR paper) Given the following assumptions about a company writing a single policy:

Earned and received premium, incurred & paid loss, incurred & paid expenses, and STAT expenses reserves follow the below pattern:

	Premium		**Losses**		**Expenses**		
Year	Earned	Received	Incurred	Paid	STAT Incurred	GAAP Incurred	Paid Expenses
0	0	75	0	0	18	0	9.0
1	100	20	72	18	12	30	13.5
2	0	5	0	36	0	0	6.0
3	0	0	0	18	0	0	1.5
4	0	0	0	0	0	0	0
Total	**100**	**100**	**72**	**72**	**30**	**30**	**30**

- Surplus is 31.5% of discounted unpaid losses.
- Losses are discounted at 6% for the purposes of surplus calculations.
- Interest rate is 6%.
- Tax rate of 35% applies to underwriting and investment income.

a. Determine the total STAT reserves for each year.

b. Determine the surplus for each year.

c. What are the investible assets for each year?

d. What is the pre-tax underwriting income for each year?

e. What is the IRR on equity flows?

Ro-30. Given the following:

Year	After-Tax Income	GAAP Equity
0	0	38.20
1	2.76	15.74
2	2.82	5.35
3	0.97	0
4	0	0

a. Determine the return indicated by the PVI/PVE measure. Use a 10.74% discount rate for income and equity.

b. Determine the return indicated by the PVI/PVE measure. Use a 12% discount rate for income and equity.

Ro-31. **(★★)** (From Exhibit 3, Sheet 1 of the IRR paper) Given the following assumptions about a company writing a single policy:

Earned and received premium, incurred & paid loss, incurred & paid expenses, and STAT expenses reserves follow the below pattern:

	Premium		**Losses**		**Expenses**		
Year	Earned	Received	Incurred	Paid	STAT Incurred	GAAP Incurred	Paid Expenses
0	0	75	0	0	18	0	9.0
1	100	20	68	18	12	30	13.5
2	0	5	3	36	0	0	6.0
3	0	0	1	18	0	0	1.5
4	0	0	0	0	0	0	0
Total	**100**	**100**	**72**	**72**	**30**	**30**	**30**

- Reserves for unpaid losses are discounted at 6% per year. Expenses are not discounted for reserve purposes.
- Surplus is 31.5% of discounted unpaid losses.
- Interest rate is 6%.
- Tax rate is 35% for underwriting and investment income.

Determine the IRR on equity flows.

Ro-32. (★★) Given the following assumptions about a company writing a single policy:

Earned and received premium, incurred & paid loss, incurred & paid expenses, and STAT expenses reserves follow the below pattern:

	Premium		Losses		Expenses & Reserves		
Year	Earned	Received	Incurred	Paid	STAT Incurred	GAAP Incurred	Paid Expenses
0	0	500	0	0	150	0	112.5
1	1,000	400	700	250	120	270	67.5
2	0	100	250	250	0	0	30
3	0	0	50	250	0	0	30
4	0	0	0	250	0	0	30
Total	**1,000**	**1,000**	**1,000**	**1,000**	**270**	**270**	**270**

- Surplus is 50% of statutory reserves.
- Investment return is 8%.
- Tax rate is 21% for underwriting and investment income.

Determine the return suggested by the PVI/PVE measure, using a 9% rate for discounting on each of income and equity.

Ro-33. **(★★)** (From Exhibit 3, Sheet 3 of the IRR paper – **Note:** This question is meant to demonstrate the results shown in the exhibit – it is a time-consuming question not meant to be representative of an (exam-type question. You should use Excel for calculations.)

Given the following assumptions about a company writing the same type of policies each year, with each following the below pattern of premium, losses, and reserves:

	Premium		**Losses**		**Expenses**		
Year	Earned	Received	Incurred	Paid	STAT Incurred	GAAP Incurred	Paid Expenses
0	0	75	0	0	18	0	9
1	100	20	68	18	12	30	13.5
2	0	5	3	36	0	0	6.0
3	0	0	1	18	0	0	1.5
4	0	0	0	0	0	0	0
Total	**100**	**100**	**72**	**72**	**30**	**30**	**30**

- STAT Reserves for the first year are based on unearned premium and statutory expense reserves. Thereafter, reserves are based on unpaid loss and expenses. [10]
- Surplus is 31.5% of discounted unpaid losses.
- Interest rate is 6%.
- Tax rate is 35% for underwriting and investment income.
- Company writes 5% more policies each year than were written in the previous year.

a. Populate the columns in the chart above for the entire book of business over the first four years, given the assumed growth rate.

b. Determine the return on equity for each of years 1 – 4.

[10] The exhibit specifies a 6% reserve discount rate, but per Robbin, STAT reserves are undiscounted; STAT income and GAAP income are discounted for purposes of PVI calculations.

Ro-34. Given the following assumptions about an insurance company's block of business:

- Premium of $500 in the first policy year.
- Policy growth of 9% per year.
- Losses paid out over the course of 5 years, with a target loss ratio of 65%.
- Variable expenses of 20% of premium paid out on policy inception.
- Fixed expenses of 10 paid out at the end of each of the first two years.
- Premium:Surplus ratio of 2:1.
- Surplus equals investible equity.
- Surplus is determined as the sum of discounted loss reserves, unearned premium reserves, and expense reserves.
- Company invests in a variety of stocks, bonds, and other asset classes.
- Average investment return of 8%.
- Tax rates vary by asset class.
- IRR was determined to be 9%.

Determine the company's ROE for a given year, assuming that the company is in the equilibrium growth phase.

Kreps (Riskiness Leverage Models)

Kre-1. What are three desirable qualities of an allocatable risk load formulation?

Kre-2. Generally, the risk load is not a coherent measure. Which property of coherence may not be satisfied based on the choice of leverage ratio?

Kre-3. Which riskiness leverage ratio form(s), of those presented in the Kreps' paper, would be good choices in each of the following situations:

a. Desire coherence

b. Concerned about entire distribution of losses

c. Only concerned with downside deviations

d. Concerned with not meeting plan, but not risk of ruin

Kre-4. One property of Kreps' risk load using the leverage model formulation is that the risk load scales with currency change; that is, $R(\lambda X) = \lambda R(X)$, given that $L(x)$ is homogeneous of order zero: $L(\lambda x) = L(x)$.

Why is the homogeneity of the leverage function required? How can $L(x)$ be constructed to satisfy this property?

Kre-5. For management, what are desirable qualities of riskiness leverage ratios? How about for regulators?

Kre-6. Suppose management has adopted a rule such that "surplus should be 2 times the average negative income in the cases where it is below the 3% level." You run several simulations and observe that one particular line (Line S) produces extremely negative income in the right tail, relative to other lines.

a. How will the mean return on surplus for Line S likely compare to the returns on the other lines? Why?

b. Based on the results from (a), management would like to reduce the underwriting volume of Line S only. Provide two reasons that this could potentially not be practical and suggest another approach that management may take to lessen the burden from Line S.

Kre-7. Suppose you are running scenarios of potential outcomes for a two-line company, using the TV@R method. You are simulating the results for various percentile levels, with the following results (from the paper):

%	Income is Below	Mean Value of TVAR and Allocation Percentages Total	Line A	Line B	Investment
0.1	(8,892,260)	(10,197,682)	12.30%	85.99%	1.71%
0.2	(7,967,851)	(9,326,936)	12.49%	85.73%	1.78%
0.4	(7,024,056)	(8,380,265)	12.89%	85.09%	2.02%
1	(5,749,362)	(7,129,796)	13.38%	84.67%	1.95%
2	*(4,732,795)*	*(6,159,564)*	*13.60%*	*84.30%*	*2.10%*
5	(3,309,641)	(4,811,947)	13.60%	84.20%	2.20%
10	(2,143,327)	(3,734,177)	13.26%	84.94%	1.80%

a. Which of the two lines is riskier? Approximately how many times more surplus must be allocated to that line according to the scenarios?

b. Suppose management decides that they desire surplus to be 1.5 times the average negative income for the worst 2% of scenarios. What is the appropriate level of surplus? (Round to the nearest million dollars.)

c. Suppose management wants to reduce the required surplus. Describe two ways they can achieve that goal.

Kre-8. Suppose an insurer's universe of outcomes is given by the following:

Event	Probability, f(x)	Loss Size, x
1	0.9	0
2	0.08	10
3	0.02	50

Determine the appropriate risk load using the given leverage ratio, and the appropriate form for $L(x)$ as indicated in Kreps' paper.

Derive the risk load using Kreps' formulation: $R = \int f(x)\,(x - \mu)L(x)dx$

a. Constant
b. Variance
c. TV@R(95%)
d. TV@R(90%)
e. V@R(95%)
f. Semi-Variance
g. Mean downside deviation

Mango (An Application of Game Theory: Property

Man-1. What does it mean when we say that the marginal surplus and marginal variance methods are order dependent? What is the concern with this property?

Man-2. Why, mathematically, does the marginal surplus method work out to be sub-additive? (Also, what does it mean to be sub-additive?)

Man-3. Why, mathematically, does the marginal variance method work out to be super-additive? (Also, what does it mean to be super-additive?)

Man-4. What does Mango mean when he references "renewal additivity"? Why is this desirable?

Man-5. What's so great about the Shapley Value Method? How does the Covariance Share Method improve upon it?

Man-6. The Shapley Value Method was shown for marginal variance. Could it be used for the marginal standard deviation method?

Man-7. Given the below for two portfolios X and Y:

- $\text{Var}(X) = 200$
- $\text{Var}(Y) = 50{,}000$
- $\text{Var}(X + Y) = 52{,}000$
- Management requires surplus at a level satisfying an exceedance probability of 2.5%
- Management requires a 12% return on surplus

Φ(z)	95.0%	97.5%	98.0%	99.0%	99.5%
z	1.645	1.960	2.054	2.326	2.576

(a) – (e): Using the Marginal Surplus method, determine the risk load required for:

a. Portfolio X, if X is added first
b. Portfolio Y, if Y is added first
c. The total of the combined portfolio
d. Portfolio X, when X is added to Y
e. Portfolio Y, when Y is added to X

f. Compare the total from (d) and (e) to your answer from (c).

Man-8. Repeat the prior exercise using the Marginal Variance method.

Man-9. Repeat the prior exercise using the Shapley Value Method.

Man-10. A firm writes two lines of business with exposure to loss from independent events as described below:

Event	Probability	LOB A	LOB B
1	0.2%	20	130
2	0.5%	50	90
3	3.0%	70	120

Determine the renewal risk load assigned to each line of business, assuming a required return of equity of 14% and z = 2.0. To save on some calculations, you are given that:

- Var(A+B) = 1,192.93
- Var(B) = 493.07

a. Marginal Surplus
b. Marginal Variance
c. Shapley Method
d. Covariance Share

Man-11. Explain whether a cooperative game would be best represented by a sub-additive or super-additive characteristic function.

Man-12. Suppose a game is started in which initially there are coalitions denoted $S_1, S_2, S_3, \ldots$. The total number of players in the game is denoted N. If the game is super-additive, determine the optimal number of coalitions.

Man-13. Three jewelry designers, Anna, Belle, and Carrie, are in a new shop, Beads 'N' Things, wherein they may purchase large quantities of beads. Bead prices are as below:

500 beads	750 beads	1,000 beads
$7	$9	$11

Beads 'N' Things' credit card machine has just gone down, and they are only accepting cash. Anna, Belle, and Carrie would like to make a purchase today, but none of them brought much cash. Anna has $4; the other two ladies have $3 each. None of the ladies care much about the small sums of cash they brought; they really want beads.

a. Show whether this situation is sub-additive or super-additive.

b. Carrie proposes that the ladies should pool their resources and purchase 750 beads – she says each of Anna and Belle can keep 200 beads, and she will take the remaining 350. She reasons that although she ends up better off, it still works for everyone since they are all better off in the group than alone.

   Does Carrie's proposal have merit?

c. Anna proposes another pool in which she keeps all 750 beads. With respect to game theory, would Anna's proposal be in the core of this game?

Man-14. **(★★)** In a game of Pokémon, four monsters, Absol, Baltoy, Cacnea, and Delibird would each like to defeat a raid boss, Eternatus. The prize for defeating this legendary dragon is 30,000 experience points (XP). XP are very coveted by all the monsters.

Suppose Eternatus requires 530 hit points (HP) to be defeated. Absol can contribute 450 HP, Baltoy can contribute 250 HP, and the other two can contribute up to 150 HP. Each of the four fighting monsters can regenerate HP after some time, so they do not care how much HP they use.

The monsters choose to band together and allocate winnings based on the average marginal contribution of each monster, when considering all possible orderings of all the four monsters.

Determine how much of the prize each monster would be allocated.

Man-15. Given the following information about three insurers, A, B, and C:

- All insurers write monoline property business, in the same regulatory jurisdiction, writing the same amount of premium.
- Each insurer holds capital of 1,000.
- Should any two insurers combine business, the required capital would be 1,700.
- Should all three insurers combine business, the required capital would be 2,500.
- A proposes combining the business with any one of the other insurers, with A contributing 750 in capital, and the other contributing the remainder.
- Neither B nor C propose a different combination scenario.

Discuss whether A's proposal is fair. Also discuss whether, should the three insurers combine, the combination would meet the conditions for individual and collective rationality.

Man-16. **(★★)** (Exam 9, 2019 Q18, Part c)

Three standalone entities are considering forming a single company to reduce required funds. The required funds for each possible entity and combination thereof are shown below.

Entity/ies	1	2	3	1&2	1&3	2&3	1&2&3
Required Funds ($M)	1.9	3.2	5.1	4.5	6.4	7.5	8.7

Determine the range of funds that could be contributed by Entity 2 and still satisfy the conditions of rationality.

OBJECTIVE D REVIEW SOLUTIONS

Ferrari (The Relationship of Underwriting, Investment, Leverage, and Exposure to Total Return on Owners' Equity)

Fer-1 Sol. If insurance exposure (P/S) increases, investment gain on assets (I/A) may decrease because the company might tend to adopt a more conservative investment strategy to offset the higher exposure.

Fer-2 Sol. Insurance exposure (P/S) and underwriting profit (U/P) may move in the same direction because as profit increases, the company will be impelled to write more business.

Fer-3 Sol. Underwriting Profit (U/P) and investment return on assets (I/A) move in the same direction because if the firm has a higher U/P, it can take on riskier investments, and potentially earn higher returns than they would under a conservative strategy.

Fer-4 Sol. Use $\frac{T}{S}=\frac{I}{A}\left(1+\frac{R}{S}\right)+\frac{U}{P}\cdot\frac{P}{S}$

$$\frac{T}{S}=7\%\left(1+\frac{3{,}000}{2{,}000}\right)+\frac{300}{4{,}000}\cdot\frac{4{,}000}{2{,}000}=\mathbf{32.5}\%$$

Alternatively, use $\frac{T}{S}=\frac{I}{A}+\frac{R}{S}\left(\frac{I}{A}+\frac{U}{R}\right)=7\%+\frac{3{,}000}{2{,}000}\left(7\%+\frac{300}{3{,}000}\right)=\mathbf{32.5}\%$

McClenahan (Insurance Profitability)

McC-1 Sol. The policyholder is paying upfront for an uncertain event. Instead of purchasing the insurance policy, he could invest the policy premium so that he could be prepared in the event of that contingency, and in case the contingency does not happen, he would retain his investment. Even though the policyholder could have invested in any other asset, his investment in the insurance company (by way of premium payment) is equivalent to a risk-free investment, in that he does not receive dividends nor does he bear the burden if the insurer's investments lose value.

McC-2 Sol. Shortcomings are:

- Rate equity is not the same as rate-of-return equity. The former is a desirable quality, while the latter is forced as a substitute if comparing ROE for purposes of regulation. Under this measure, companies that are otherwise identical except for the amount of surplus held would need to charge different rates for the same exposure to risk.
- Allocation of equity is a subjective process.
- Return on sales is an improvement because it relates profit to premium, which is a natural interpretation of profit.

McC-3 Sol.

Time	Premium	Loss	Expenses	Total Cash Flow
0	100,000			100,000
0.5		(16,250)	(35,000)	(51,250)
1.5		(22,750)		(22,750)
2.5		(13,000)		(13,000)
3.5		(7,800)		(7,800)
4.5		(5,200)		(5,200)

The net present value of the cash flows, at 6%, is 7,776 (or 7.78% of premium). This represents the opportunity cost expected to be suffered by the average policyholder for the risk-free income lost by taking out an insurance policy rather than investing the money to pay off a potential loss.

Per the paper, note that this *does not* represent the expected insurer profit, which should be greater than the risk-free rate.

McC-4 Sol. Using rate equity, companies A & B would have the better proposals, since they offer lower rates for the same expected loss distribution. This is intuitively the "better" measure of the concept of equity.

Regulators typically measure the return on equity, which is as below for each company:

Company	Equity	Return	ROE
A	$315 \div 3 = 105$	$315 - 300 = 15$	$15 \div 105 = 14.3\%$
B	$315 \div 1.5 = 210$	$315 - 300 = 15$	$15 \div 210 = 7.1\%$
C	$330 \div 3 = 110$	$330 - 300 = 30$	$30 \div 110 = 27.2\%$
D	$330 \div 1.5 = 220$	$330 - 300 = 30$	$30 \div 220 = 13.6\%$

By an ROE measure, companies B and D would be preferred over A and C, which is a troublesome inconsistency based on the intuitively more preferable rate equity concept.

Feldblum (IRR)

IRR-1 Sol.

During Policy Period Only

- Pricing risk (adverse experience)
- Catastrophe risk

Until Losses are Paid

- Asset risk
- Reserving risk
- Asset-liability mismatch
- Credit risk (collection of premiums or reinsurance)

IRR-2 Sol.

1. IRR has a "presentation problem." Although an IRR may be positive, and higher than the return on investment, it still suggests an unprofitable project if it is lower than the cost of capital. This may be difficult to explain to a regulator.
2. Related to the presentation problem is the idea of selecting between mutually exclusive projects based on IRR alone. A higher IRR will make a project appear more favorable, but does not consider that the actual cash in dollar terms may be lower.
3. IRR may have ambiguous results. In the case that signs change more than once, the IRR may have multiple solutions. This is not really applicable when using IRR on equity flows, because generally there will only be one sign change from negative to positive flows.
4. IRR implicitly assumes that a project can be invested at the rate implied by the IRR, which is not necessarily true. However, this issue is not necessarily applicable to insurance companies, since they can indeed just write more business.
5. IRR calculations frequently are simplified for ease of use. The many assumptions that are used may result in an inappropriate indication.

IRR-3 Sol. Assume the premium is 100, so expenses are 20, and losses are 80 (40 in Year 1, 40 in Year 2).

Get Equity Flows as shortfall between assets required and cash.

		Year 0	Year 1	Year 2
Cash	BOY Assets	-	128.4	64.2
	+ Premium	100	-	-
	- Paid Loss & Expense	-20	-40	-40
	Total Cash	80	88.4	24.2
Assets	Reserves	80	40	0
	Surplus (based on ratio)	40	20	0
	Total Assets	120	60	0
	EOY Assets (after inv. Income)	128.4	64.2	0
	Equity Flow (Cash – Assets)	-40	28.4	24.2

IRR is solved by: $0 = -40 + \frac{28.4}{1+IRR} + \frac{24.2}{(1+IRR)^2} \rightarrow IRR = \mathbf{21.0}\%$

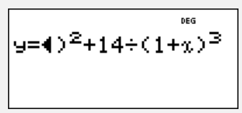
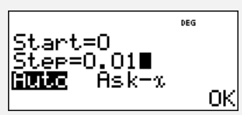
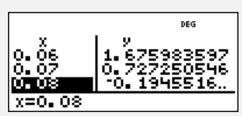
IRR-4 Sol.

		Year 0	Year 1	Year 2	Year 3
Cash	BOY Assets	0	161.33	102.67	44
	+ Premium	100	0	0	0
	- Paid Loss & Expense	-10	-40	-40	-30
	Total Cash	90	121.33	62.67	14
Assets	Reserves	110	70	30	0
	Surplus (based on ratio)	36.67	23.33	10	0
	Total Assets	146.67	93.33	40	0
	EOY Assets (after inv. Income)	161.33	102.67	44	0
	Equity Flow (Cash – Assets)	-56.67	+28	+22.67	+14

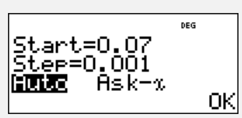
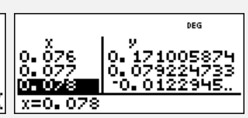
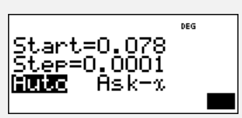
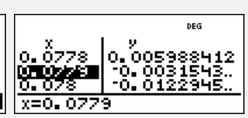
Solve the resulting equation to get **IRR = 7.79%**

$$-56.67 + \frac{28}{1 + IRR} + \frac{22.67}{(1 + IRR)^2} + \frac{14}{(1 + IRR)^3} = 0$$

Note: To solve this equation, I use guess-and-check on my calculator (TI-30XS Multiview). Process is below:

<table>
<tr><td>1. Enter left-side of equation in table  and hit enter.</td><td></td></tr>
<tr><td>2. Start at 0 and step by 0.01; arbitrary, but reasonable for an IRR question; for IRR we often get an answer around 10% or so.</td><td></td></tr>
<tr><td>3. Scroll down a bit to see where y changes signs. This tell us that answer is between 0.07 and 0.08</td><td></td></tr>
<tr><td>4. Iterate steps 2 and 3 until you get something pretty close to 0:

Go back to table function, hit “enter” to get to start/step, and then adjust.

7.79% yields an answer pretty close to 0.</td><td></td></tr>
</table>

On an exam, you probably wouldn’t need to know how to solve an equation of this type, but the process above can be used to solve any equation.

For purposes of these exercises, where I use *a lot* of these types of higher-order equations, if you don’t feel like using the calculator, you can use the IRR function in Excel, which would look like:

	A	B	C	D
1	-56.67	28	22.67	14
2	7.79%	=IRR(A1:D1,0.1)		

IRR-5 Sol. The setup is identical to the previous problem. We need to modify the surplus row, which affects required assets.

		Year 0	Year 1	Year 2	Year 3
Cash	BOY Assets	0	181.5	115.5	49.5
	+ Premium	100	0	0	0
	- Paid Loss & Expense	-10	-10 – 30	-40	-30
	Total Cash	90	141.5	75.5	19.5
Assets	Reserves	110	70	30	0
	Surplus (based on ratio)	55	35	15	0
	Total Assets	165	105	45	0
	EOY Assets (after inv. Income)	181.5	115.5	49.5	0
	Equity Flow (Cash – Assets)	-75	+36.5	+30.5	+19.5

This time, we get **IRR = 8.35%**. This demonstrates that IRR decision making can be sensitive to the method of surplus allocation – if the cost of capital were set at 8%, the firm would accept the business using this allocation of surplus but would reject it using the 3:1 allocation from the previous problem.

IRR-6 Sol. Now we have the following:

		Year 0	Year 1	Year 2	Year 3
Cash	BOY Assets	0	181.5	137.5	93.5
	+ Premium	100	0	0	0
	- Paid Loss & Expense	-10	-10 – 30	-40	-30
	Total Cash	90	141.5	97.5	63.5
Assets	Reserves	110	70	30	0
	Surplus (held until losses paid)	55	55	55	0
	Total Assets	165	125	85	0
	EOY Assets (after inv. Income)	181.5	137.5	93.5	0
	Equity Flow (Cash – Assets)	-75	+16.5	+12.5	+63.5

This time, we get **IRR = 8.82%**. The firm would still accept the business under this scenario, but note that the IRR did change, so we could potentially have a different decision if the cost of capital were different. The point here is to again demonstrate IRR's sensitivity to the timing of surplus release.

IRR-7 Sol.

a. Since all lines write $1,000 in premium, we would allocate evenly ($333.33) to each.

b. For line A, the average reserve is (1000)(0.80)(0.5) = 400. Likewise, for line B, we have 700, and 3000 for line C. Our allocation becomes:

- $S_A = 1000\left(\frac{400}{400+700+3000}\right) = \$97.56$
- $S_B = 1000\left(\frac{700}{400+700+3000}\right) = \$170.73$
- $S_C = 1000\left(\frac{3,000}{400+700+3000}\right) = \$731.71$

IRR-8 Sol. The surplus allocated to A is $1,500\left(\frac{2000}{2000+1000}\right) = 1,000.$

Determine equity flows. (Note that since the average time between claim occurrence and payment is 0.5 years, and the average time between the policy start and claim occurrence is 0.5 years, losses on average are paid at time t = 1.)

		Year 0	Year 1
Cash	BOY Assets	0	2,782
	+ Premium	2,000	0
	- Paid Loss & Expense	-400	-1,600
	Total Cash	1,600	1,182
Assets	Reserves	1,600	0
	Surplus	1,000	0
	Total Assets	2,600	0
	EOY Assets (after inv. Income)	2,782	0
	Equity Flow (Cash – Assets)	-1,000	+1,182

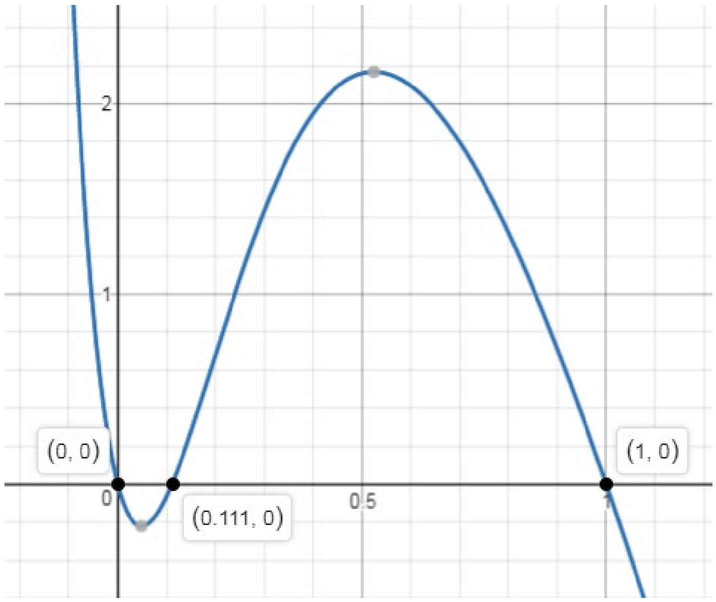
Solve $-1000 + \frac{1182}{1+IRR} = 0 \rightarrow IRR = \mathbf{18.2\%}$

IRR-9 Sol. Surplus is meant to cover the business in case of adverse deviation. Many factors can influence the level of risk a line has, including the volume of premium written, but also:

- Long-tail versus short-tailed lines: As long-tailed lines are riskier, they have greater reserves and should be allocated relatively more surplus.
- Occurrence versus claims-made contracts: Since claims-made contracts have less uncertainty, they should be assigned less surplus.
- Service contracts: These contracts have no insurance risk, so they should have relatively less surplus.
- Retrospective versus prospective: Retrospective contracts are more shielded from insurance risk than are prospective, so this is another thing to consider when assigning surplus.

IRR-10 Sol.

a. $-45 + \frac{140}{1+IRR} - \frac{55}{(1+IRR)^2} - \frac{140}{(1+IRR)^3} + \frac{100}{(1+IRR)^4} \rightarrow IRR \in \{0, 11.1\%, 100\%\}$



b. This exemplifies the property that IRR may yield ambiguous results (if the cash flows change signs more than once).

c. In insurance equity flows, the cash flows change only from negative (when capital required) to positive (when capital is released as losses are paid).

IRR-11 Sol.

a. We solve to get an IRR of **5.57%,** based on the below.

		Year 0	Year 1	Year 2	Year 3
Cash	BOY Assets	0	8,662.50	5,250	2,625
	+ Premium	5000	0	0	0
	- Paid Loss & Expense	-450	-450-1500	-1500	-1500
	Total Cash	4,550	6,712.50	3,750	1,125
Assets	Reserves	4,950	3,000	1,500	0
	Surplus	3,300	2,000	1,000	0
	Total Assets	8,250	5,000	2,500	0
	EOY Assets (after inv. Income)	8,662.50	5,250	2,625	0
	Equity Flow (Cash – Assets)	-3,700	1,712.50	1,250	1,125

b. The company says that the premium is insufficient because the IRR of 5.57% is less than the cost of capital (6%). The regulator may overlook this because the IRR is positive and moreover, greater than the investment return, so a quick glance would make it seem fine. This is intended to exemplify the "presentation problem" of IRR.

Robbin (The Underwriting Profit Provision & IRR, ROE, and PVI/PVE)

Rob-1 Sol. The company may want to write the business in order to maintain relationship with the client. The company may also choose to write at a loss because investment profit may offset the underwriting loss, making the contract profitable in any case.

Rob-2 Sol. The measurement basis differs – profit targets are based on prospective policy year basis, while realized profit is usually measured on a calendar year basis. Also, not all investment income "belongs" to the policyholders – some investment income comes from stockholders-supplied funds, so should not be considered as part of policyholder profit.

Rob-3 Sol. The paper mentions:

- Underwriting profit – used in manual rates and rate filings
- Corporate target underwriting profit provisions – used to set premiums
- Breakeven underwriting profit provisions – based on targeting a risk-free rate of return for stockholders
- Charged profit provision – rate charged after adjusting manual rate for experience, schedule rating, and other adjustments
- Actual underwriting profits – calculated *ex post*, used to consider appropriateness of estimates

Rob-4 Sol. The IRR paper give three ways to measure return:

- Investor return on equity
- Apply the corporate return to a policy, e.g., using GAAP ROE
- Extend CY ROE measure to the life of a policy

Rob-5 Sol. For utility companies, measuring rate of return is fairly simple since there is no need to allocate surplus based on risk. For insurance companies, surplus is allocated based on the varying levels of risk between lines of business, and this process convolutes things, as it is artificial and arbitrary.

Rob-6 Sol. By considering equity, these methods implicitly include the impact from the accounting treatment of expenses and surplus, which is ignored in the underwriting cash flow methods.

Rob-7 Sol. There are many reasons.

- The RADCF method determines a fair premium directly, while the PVI/PVE method determines an indicated premium need to hit a target return on surplus. The methodologies are inherently different.
- The RADCF method makes no direct use of surplus, other than for purposes of determining investment income on investible surplus. This generally has a very small impact on the indications.
- Because RADCF does not make use of surplus, there is no underlying accounting structure – it does not consider the conservative treatment of acquisition expenses under statutory accounting.
- The RADCF risk-adjusted rate is fairly subjective; there is no generally accepted method to determine what is reasonable here.

Rob-8 Sol.

- External factors (regulatory requirements, market competitiveness)
- Whether and how surplus should be reflected
- Whether and how risk should be reflected
- Whether cash or income flows are used
- How to reflect taxes

Rob-9 Sol.

a. $U = U_0 - i_{AT}PHSF$
   U = final profit

   U_0 = traditional profit

   i_{AT} = after-tax investment income

   $PHSF$ = policyholder supplied funds

b. PHSF are subtracted out to reflect that premiums generate only a portion of invested assets (so not all the profit from invested assets belongs to the policyholder). PHSF takes the sum of:

   - UPR, net of prepaid acquisition expenses, since those funds are not available to invest, and net of premiums receivable, since those funds are also not available to invest, and
   - Portion of reserves that are investible $\left(PLR \times (\text{Reserves} \div \text{Loss})\right)$

c. This method uses CY data, which are easily attainable. Also, except in periods of rapid growth or decline, CY investment income is fairly stable. This method would not be appropriate for periods with rapid growth or decline.

Rob-10 Sol.

a. $U = U_0 - PLR\left(PV(x_{\text{reference}}) - PV(x_{\text{reviewed}})\right)$

- U = final profit
- U_0 = traditional profit
- PLR = permissible loss ratio
- x_i = cash flows for respective line i

b. You can consider the new money yield (in line with prospective ratemaking), or portfolio yield from recent history (which is less appropriate for prospective ratemaking, but easier to verify).

c. You can account for taxes in part by using an after-tax interest rate for discounting. The after-tax interest rate can be determining on a prospective or retrospective basis.

d. An advantage here is that, unlike the CY Offset method, there is no distortion in periods of rapid growth or decline. A disadvantage is that it requires determination of an appropriate discount rate.

Rob-11 Sol.

a. Formula: $\text{PV}(\Delta\text{Equity}; r_r) = \text{PV}(\text{CF}; r_i)$

- ΔEquity is discounted at the target rate of return
- Indicated premium is that which makes the equation hold.
- CF are discounted at an investment rate of return, and includes cash flows from underwriting income, investment income, and income taxes

b. Disadvantages include need to allocate surplus and to estimate future equity flows. Also, this method does not reconcile to GAAP easily because of use of equity flows versus underwriting flows. But it does make sense intuitively.

Rob-12 Sol.

a. Formula: $\mathrm{PV}(\mathrm{Premium}; \mathrm{r_f}) = \mathrm{PV}(\mathrm{Expenses} + \mathrm{Taxes}; \mathrm{r_f}) + \mathrm{PV}(\mathrm{Losses}; \mathrm{r_A})$

- Premium, expenses, and taxes are discounted at the risk-free rate
- Losses are adjusted at a risk-adjusted rate

b. Losses use a lower risk-adjusted rate because they are less certain than the other cash flows.

c. Robbin uses the CAPM method, which has several pitfalls, including some mentioned in the BKM text. For one, it implicitly assumes that all risk is based on systematic risk, and does not consider other factors, so the theory is questionable. Also, CAPM betas are difficult to estimate in the first place, even if the theory were perfect. Lastly, even if CAPM betas were easy to estimate for stocks, it's more complex trying to break it down to an individual line of business.

d. A key advantage here is again that it's intuitively a "correct" measure of profit. Also, it does not rely heavily on surplus (only needed for consideration of investment income on surplus). The disadvantage is the sensitivity of the result to the discount rate.

Rob-13 Sol.

a. The IRR method solves for the discount rate so that the sum of the discounted equity flows from time 0 until policy expiry is 0. The policy is acceptable if it generates an IRR at least as high as the target.

b. In general, a disadvantage of IRR is the potential for multiple roots. This is not applicable when using IRR for equity flows since equity flows generally change sign only once (negative on influx of capital, and then positive). The other disadvantage that remains even when using equity flows is that the IRR model implicitly assumes that funds can be reinvested at the IRR, which is not necessarily the true cost of capital.

Rob-14 Sol.

a. $\frac{PVI}{PVE} = (1 + r_I) \sum_{j=1}^{n} \frac{I_j}{(1+r_I)^j} \Big/ \sum_{j=0}^{n-1} \frac{Q_j}{(1+r_Q)^j}$. We are taking the ratio of present value of income (as of end of year) to present value of equity (at beginning of year). Each can be discounted at different rates, but generally using the cost of capital for both is pretty sensible.

b. Explicit use of the discount rate of cost of capital overcomes the IRR issue about not being able to reinvest at the rate. A disadvantage is that this method is not easily determined for non-annual cash flows, due to its use of equity.

Rob-15 Sol.

a. CY ROE method is a standard return on equity calculation, where income and equity are discounted at the growth rate. The formula applies when the company is in the equilibrium growth phase.

b. ROE is well-known and understood, which is good.

Rob-16 Sol.

	a. Traditional Load	b. CY Data	c. Equity	d. Discount Rate
CY Inv Offset	**Yes**	**Yes**	No	No
PV Offset	**Yes**	No	No	**Yes**
PVCF	No	No	**Yes**	**Yes**
RADCF	No	No	No	**Yes**
IRR	No	No	**Yes**	No
PVI/PVE	No	No	**Yes**	**Yes**
Growth CY ROE	No	**Yes**	**Yes**	No

Rob-17 Sol. PHSF are the investible assets, which come from premium and reserves.

The portion from premium left to invest, as a portion of earned premiums, is:

$$\frac{300{,}000 - 0.3 \cdot 80{,}000 - 40{,}000}{700{,}000} = 33.7\%$$

The above uses unearned premiums of 1,000,0000 – 7,000,000 = 300,000, and an unearned premium ratio of 300,000 ÷ 1,000,000 = 0.3.

The portion from reserves available to invest, as a portion of earned premium is:

$$60\% \times \frac{750{,}000}{700{,}000} = 64.28\%$$

PHSF is the sum: **98%.**

Rob-18 Sol. First, need PHSF, composed of investible funds from loss reserves and net unearned premiums reserves:

- Investible UPR: $\frac{30{,}000\times(1-15\%)-10{,}000}{80{,}000} = 19.375\%$
- Investible Reserves: $1.0 \times 0.7 = 70\%$
- PHSF: 89.375%
- Indicated Profit Provision: $U = U_0 - i_{AT}PHSF = 5\% - 8.9\%(89.375\%) = \mathbf{-2.95}\%$

Rob-19 Sol. First, need PHSF, composed of investible funds from loss reserves and net unearned premiums reserves:

- Investible UPR: $\frac{70{,}000\times(1-20\%)-20{,}000}{100{,}000} = 36\%$
- Investible Reserves: $0.65 \times 0.7 = 45.5\%$
- PHSF: 81.5%
- Indicated Profit Provision: $U = U_0 - i_{AT}PHSF \rightarrow 4.8\% = 5\% - i_{AT}(81.5\%) \rightarrow i_{AT} = \mathbf{0.245}\%$

Rob-20 Sol. $U = U_0 - PLR\big(PV(X_{\text{reference}}) - PV(X_{\text{reviewed}})\big) =$
$5\% - 70\%(0.865 - 0.872) = \mathbf{5.49}\%$

Rob-21 Sol. With $v = \frac{1}{1.0625}$,

$$PV(X_{\text{reference}}) = 0.3v^{0.5} + 0.2v + 0.4v^{1.5} + 0.1v^2 = 0.93309$$

$$PV(X_{\text{reviewed}}) = 0.45v^{0.5} + 0.1v + 0.45v^{1.5} = 0.9416$$

$$U = U_0 - PLR\big(PV(X_{\text{reference}}) - PV(X_{\text{reviewed}})\big) = \mathbf{3.05}\%$$

Rob-22 Sol.

a. $(1-34\%)(103.22 - 57.34 - 40.19) = \$\mathbf{3.76}$

b. $(1-34\%)(2.70) = \$\mathbf{1.78.}$ Note that the sum of the underwriting and investment incomes = \$5.54, which is the present value of equity changes.

c. In the present value cash flow method, equity is discounted at the target rate, while the remaining cash flows are discounted at an investment rate of return.

Rob-23 Sol.

a. The PVCF model looks for this equality: $\mathrm{PV}(\Delta\mathrm{Equity}; r_r) = \mathrm{PV}(\mathrm{CF}; r_i)$. We need underwriting cash flows and changes in equity.

Time	Loss	Expense	U/W CF	Surplus[1]	Inv Inc[2]	Equity[3]	ΔEquity
0	0	15%P	0.85P	0.5P	0	0.65P	0.65P
0.5	300	0	-300	0.5P	0.0172P	0.65P	0
1	400	50	-450	0.5P	0.0172P	0.65P	0
1.5	500	0	-500	0.5P	0.0172P	0.65P	0
2	200	0	-200	0	0.0172P	0	-0.65P
3	100	0	-100	0	0	0	0
Total	1,500	0.15P + 50	0.85P - 1550		0.0688P		0

[1] Surplus is indicated at a 2:1 premium-to-surplus ratio and is held for 2 years.

[2] Investment income each semi-annum = $(\mathrm{Surplus})(1.07^{0.5} - 1)$

[3] Equity-to-surplus ratio = 1.3:1

- PV(UW CF), using a 7% discount rate: 0.85P – 1418.65
- PV(After-Tax UW CF) = $(1 - 35\%)(0.85P - 1418.65) = 0.5525P - 922.12$
- PV(After-tax Investment Income)$_{\text{DR = 7\%}}$ = $(1 - 35\%)(0.0633P) = 0.04113P$
- PV(ΔEquity)$_{\text{DR = 15\%}}$ = 0.1585P

Setting equal, we have: $0.5525P - 922.12 + 0.04113P = 0.1585\mathrm{P} \rightarrow \boldsymbol{P = 2{,}119.20}$

The profit provision is $\frac{\sum \mathrm{UW\ CF}}{\mathrm{Premium}} = \frac{0.85 \cdot 2{,}119.20 - 1{,}550}{2{,}119.20} = \mathbf{11.9\%}$

I'm using discounting here to BOY, consistent with Robbin's exhibit.

b. Basing from the above, PV(After-Tax UW CF) = $(1 - 21\%)(0.85P - 1418.65) = 0.6715P - 1120.73$

PV(After-tax Investment Income)$_{\text{DR = 7\%}}$ = $(1 - 21\%)(0.0633P) = 0.04999P$

Then, we solve: $0.6715P - 1120.73 + 0.04999P = 0.1585\mathrm{P} \rightarrow \boldsymbol{P = 1{,}990.69}$

The profit provision is $\frac{\sum \mathrm{UW\ CF}}{\mathrm{Premium}} = \frac{0.85 \cdot 1{,}990.69 - 1{,}550}{1{,}990.69} = \mathbf{7.1\%}$

Rob-24 Sol. The PVCF model looks for this equality: $\text{PV}(\Delta\text{Equity}; r_r) = \text{PV}(\text{CF}; r_i)$. We need underwriting cash flows and changes in equity.

Time	U/W CF	Reserves	Surplus[1]	Inv Inc[2]	Equity[3]	ΔEquity
0	1240 – 200 = 1,040	1,050	525	0	525	525
1	-50 – 500 = -550	500	250	26.25	250	-275
2	-500	0	0	12.5	0	-250

[1] Surplus is half of reserves

[2] Investment income is 5% of the beginning surplus (investible equity)

[3] Equity = surplus

We want:

$$(1040 + 0) + \frac{-550 + 26.25}{1.05} + \frac{-500 + 12.5}{1.05^2} = 525 - \frac{275}{1 + r} - \frac{250}{(1 + r)^2}$$

$$0 = 425.99 - \frac{275}{1 + r} - \frac{250}{(1 + r)^2} \rightarrow \boldsymbol{r = 15.41\%}$$

Rob-25 Sol.

If the RADCF is valid here, then Premium = Loss + Expenses + FIT, at time t = 1. We need to bring the undiscounted values to the end of the first year.

- Losses use risk-adjusted discount rate: $8\% - 0.750(10.50\% - 8\%) = 6.125\%$
- Everything else uses discount rate of 8%.

To determine present value as of time t = 1, we can use average discount factor.

- Premium has an average factor of 1.08.
- Loss has an average factor of 1.019 = $0.5(1.06125) + 0.3(1) + 0.2(1.06125)^{-1}$
- Expense: 1.041 = $0.6(1.08) + 0.3(1) + 0.1(1.08)^{-1}$
- FIT on Surplus: 0.963 = $0.5(1 + 1.08^{-1})$

Apply these discount factors to our RADCF equation components.

- Let $x =$ undiscounted expenses
- ∴ undiscounted losses = $2x$
- Undiscounted premium is solved with profit provision: $-3.60\% = 1 - \frac{x+2x}{P} \rightarrow P = \frac{3x}{1.036}$

At t = 1:

- Premium = $\frac{3x}{1.036}(1.080) = 3.127x$
- Loss = $2x(1.019) = 2.038x$
- Expenses = $x(1.041) = 1.041x$
- FIT on UW = $34\%(3.127x - 2.038x - 1.041x) = 0.017x$
- FIT on Inv Inc = $1(0.963) = 0.963$

Balancing the equation:

- $3.127x = 2.038x + 1.041x + 0.017x + 0.963 \rightarrow x = 30$. (Undiscounted expense)
- Undiscounted loss = $2x$ =60
- Undiscounted premium = $\frac{3x}{1.036} = \mathbf{86.87}$

As a check, $1 - \frac{60+30}{86.87} = -3.60\%$, as desired.

Rob-26 Sol. We determine the risk-free rate by backing out from CAPM: $r_{XYZ} = r_f + \beta_{XYZ}(R_M) \rightarrow 15\% = r_f + 1.3(7\%) \rightarrow r_f = 5.9\%$

The risk-free rate is used to discount all non-loss related cash flows.

We use the liability beta to determine the risk-adjusted rate appropriate for discounting losses: $r_A = r_f + \beta_L(R_M) \rightarrow r_A = 5.9\% - 0.75(7\%) = 0.65\%$

Year	Premium	Expenses	Losses	Reserves	Surplus[1]	Inv Income[2]
0	0.5P	0.15P	0	10500	5250	0
1	0.5P	500	1000	9000	4500	315
2	0	0	4000	5000	2500	270
3	0	0	4000	1000	500	150
4	0	0	1000	0	0	30
Total	P	0.15P + 500	10,000			n/a
DR	5.9%	5.9%	0.65%	n/a		5.9%
PV[3]	1.0295P	0.15885P + 500	9,903.42			728.97

[1] Surplus = 50% of reserves

[2] Investment Income is based on 6% return on surplus (investible equity)

[3] Present Value as of Time T = 1

Present value of premium should equal present value of sum of expenses, losses, and taxes:

$$1.0295P = (0.15885P + 500) + (9{,}903.42) + 21\%(1.0295P - 0.15885P - 500 - 9{,}903.42) + 15\%(728.97)$$

$$P = 12{,}108$$

The underwriting profit provision is then: $1 - \left[15\% + \frac{10{,}000+500}{12{,}108}\right] = \mathbf{-1.72\%}$

Rob-27 Sol. **(★★)** We determine equity flows as income – change in surplus.

Income is comprised of after-tax investment income and underwriting income.

Year	Premium	Earned Premium	Losses	Expenses	Surplus	Reserves[1]	Assets[2]
0	1260	0	0	201.60	420	1260	1680
1	0	420	500	15	420	840	1260
2	0	420	300	15	420	420	840
3	0	420	200	15	0	0	0
Total		1260	1000	246.60			

Year	After-tax Inv income[3]	After tax u/w income[4]	Total Income	Change in Surplus	Equity Flow
0	0	-159.26	-159.26	420	-579.26
1	88.20	-75.05	13.15	0	13.15
2	66.15	82.95	149.10	0	149.10
3	44.10	161.95	206.05	-420	626.05
Total	198.45	10.59	209.04	0	209.04

Solve for IRR as:

$$0 = -576.26 + \frac{13.15}{1 + IRR} + \frac{149.10}{(1 + IRR)^2} + \frac{626.05}{(1 + IRR)^3} \rightarrow \boldsymbol{IRR = 11.8\%}$$

[1] Reserves only include unearned premium since losses and expenses are incurred on payment.

[2] Assets are the sum of surplus and reserves.

[3] After-tax investment income in year 1: $(1 - 25\%)(0.07)(1680) = 88.20$

[4] After-tax underwriting income in year 1: $(1 - 21\%)(420 - 500 - 15) = -75.05$

Rob-28 Sol. Determine equity flows as income – change in surplus.

Income is comprised of after-tax investment income and underwriting income.

Year	Paid Premium	Earned Premium	Incurred Losses	Paid Losses	Expenses	Reserves	Surplus[1]	Assets[2]
0	P	P	1,250	0	0.2P	1,250	833.33	2083.33
1	0	0		900	0	350	233.33	583.33
2	0	0		50	0	300	200	500
3	0	0		300	0	0	0	0
Total		P		1,250	0.2P			

Year	After-tax Inv income[3]	After tax u/w income[4]	Total Income	Change in Surplus	Equity Flow
0	0	-987.5 + 0.63P	-987.5 + 0.63P	833.33	-1,820.33 + 0.63P
1	131.67		131.67	-600	731.67
2	36.87		36.87	-33.33	70.20
3	31.60		31.60	-200	231.60

[1] Surplus = Reserves / 1.5

[2] Assets = Surplus + Reserves

[3] After-tax investment income in year 1: $(1-21\%)(0.08)(2083.33) = 131.67$

[4] After-tax underwriting income in year 0: $(1-21\%)(P-1250-0.2P) = -987.5 + 0.63P$

Solve for IRR as:

$$0 = -1{,}820.33 + 0.63P + \frac{731.67}{1.14} + \frac{70.20}{1.14^2} + \frac{231.60}{1.14^3} \rightarrow P = 1{,}532.72$$

The profit provision is: $1{,}532.72 = \frac{1{,}250}{1-20\%-\text{profit}} \rightarrow \text{Profit Provision} = -\mathbf{1.55}\%$

(Here, profit provision is low because premium was earned fully on inception, so generates a relatively large portion of present value.)

Rob-29 Sol.

a. Reserves in year 0 are the 100 in unearned premium plus 9 in stat reserve. Reserves in each year afterward are the sum of unpaid loss and expense for the remaining years. (Shown in table below.)

b. Determine the present value of unpaid losses by discounting at a rate of 6%. Then take 31.5% of the value. The present value is as of the year for which surplus is being determined.

$$S_0 = 31.5\%\left(\frac{18}{1.06} + \frac{36}{1.06^2} + \frac{18}{1.06^3}\right) = 20.20$$

$$S_1 = 31.5\%\left(\frac{36}{1.06} + \frac{18}{1.06^2}\right) = 15.74$$

$$S_2 = 31.5\%\left(\frac{18}{1.06}\right) = 5.35$$

c. Investible assets for each year are Surplus + Reserves – Premium Receivables

d. GAAP underwriting income is earned premium – incurred loss – GAAP incurred expense.

e. Determine equity flows as **income – change in equity**. Equity at policy inception also includes DAC, which was already recognized in the STAT incurred expense.

After-tax income in year 0 is: $(1 - 35\%)(0) = 0$

After-tax income in year 1 is: $(1 - 35\%)(6\%(104.20) - 2) = 2.76$

After-tax income in year 2 is: $(1 - 35\%)\big(6\%(72.24)\big) = 2.82$

Year	STAT Reserves	Surplus	Invested Assets	Underwriting Income	GAAP Equity
0	109	20.20	104.20	0	38.20
1	61.5	15.74	72.24	-2	15.74
2	19.5	5.35	24.85	0	5.35
3	0	0	0	0	0
4	0	0	0	0	0

Year	After-Tax Income	Change in Equity	Equity Flow
0	0	38.2	-38.2
1	2.76	-22.46	25.22
2	2.82	-10.39	13.21
3	0.97	-5.35	6.32
4	0	0	0

IRR solves:

$$0 = -38.2 + \frac{25.22}{1 + IRR} + \frac{13.21}{(1 + IRR)^2} + \frac{6.32}{(1 + IRR)^3} \rightarrow \boldsymbol{IRR = 10.74\%}$$

Rob-30 Sol.

a. PVI/PVE determines present value, where income is discounted to year 1, and equity to year 0. We have, when v = 1/1.1074,

$$\text{PVI/PVE} = \frac{2.76 + 2.82v + 0.97v^2}{38.20 + 15.74v + 5.35v^2} = \mathbf{10.74\%}$$

Note: We saw in the previous problem that used the same income and equity that IRR is 10.74%. We would thus expect PVI/PVE measure to be the same if we use the discount rate = IRR.

b. If we let v = 1/1.12, PVI/PVE = **10.71%**

Rob-31 Sol.

Year	Reserves[1]	Surplus[2]	Invested Assets[3]	Underwriting Income[4]	GAAP Equity[5]
0	109	20.20	104.20	0	38.20
1	57.5	15.74	68.23	2	15.74
2	18.5	5.35	23.83	-3	5.35
3	0	0	0	-1	0
4	0	0	0	0	0

Year	After-Tax Income[6]	Change in Equity	Equity Flow[7]
0	0	38.20	-38.20
1	5.36	-22.46	27.82
2	0.71	-10.40	11.11
3	0.28	-5.35	5.63
4	0	0	0

[1] Reserves in year 0 are 100 in unearned premium + 9 in stat reserve; thereafter reserves = sum of discounted unpaid losses and expenses.

$$\text{Reserves}_1 = \frac{36}{1.06} + \frac{18}{1.06^2} + 7.5 = 57.5$$

[2] Sample Calc for Year 1 surplus: $S_1 = 31.5\%\left(\frac{36}{1.06} + \frac{18}{1.06^2}\right) = 15.74$

[3] Invested assets are reserves + surplus – premiums receivable

[4] (Pre-Tax) Underwriting income is earned premium – incurred losses – GAAP incurred expenses

[5] GAAP Equity = Surplus, except in the first year, which also includes DAC (= year 0 expenses in STAT)

[6] For year 1: $(1 - 35\%)\big(2 + 6\%(104.20)\big) = 5.36$

[7] Equity flow is (after tax income) – (change in equity)

IRR solves: $0 = -38.2 + \frac{27.82}{1+IRR} + \frac{11.11}{(1+IRR)^2} + \frac{5.63}{(1+IRR)^3} \rightarrow \boldsymbol{IRR = 10.98\%}$

Rob-32 Sol.

- Invested Assets = Reserves + Surplus – Premium Receivables
- Investment Income = 8% of Invested Assets
- GAAP UW Income is Earned Premium – Incurred Loss – Incurred Expense
- GAAP Equity = Surplus generally, except in Year 0, where GAAP Equity = Surplus + DAC.
- After-tax income is 79% of the sum of investment income (8% on assets) and underwriting income.

Year	STAT Reserves	Surplus	Investible Assets	AT Invest Inc	UW Inc	Equity	AT Income
0	1,037.50	518.75	1,056.25		0	668.75	0.00
1	540	270	710	66.76	30	270	90.46
2	510	255	765	44.87	-250	255	-152.63
3	280	140	420	48.35	-50	140	8.85
4	0	0	0	26.54	0	0	26.54

<u>Sample Calcs (Year 2)</u>

- STAT Reserve = (Cumulative) Incurred Loss – (Cumulative) Paid Loss + Expense Reserve = $(700+250)-(250+250)+(60)=510$
- STAT Surplus = $50\%(510)=255$ (same as GAAP Equity)
- Investible Assets = $510+255=765$
- After-Tax Investment Income = $710(8\%)(1-21\%)=44.87$
- UW Income = (Year 2) Earned Premium – (Year 2) Incurred Loss = $0-250=-250$
- After-Tax Income = $44.87+(1-21\%)(-250)=-152.63$

- Determine PVI/PVE using a discount rate of 9% for each component. Income is discounted to the end of the first year; equity to the beginning of the first year. This gives us:

$$\frac{90.46-152.63v+8.85v^2+26.54v^3}{668.75+270v+255v^2+140v^3}=\mathbf{-1.745\%}$$

Rob-33 Sol. (Really long)

a. We sum up policies written at the beginning of the 0 – 3rd years.

Policy Beginning in Year 0

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
0	0	75	0	0	18	0	9	9.0
1	100	20	68	18	12	30	13.5	7.5
2	0	5	3	36	0	0	6.0	1.5
3	0	0	1	18	0	0	1.5	0
4	0	0	0	0	0	0	0	0

Policy Beginning in Year 1 (values shifted down and increased by 5%)

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
1	0.00	78.75	0.00	0.00	18.90	0.00	9.45	9.45
2	105.00	21.00	71.40	18.90	12.60	31.50	14.18	7.88
3	0.00	5.25	3.15	37.80	0.00	0.00	6.30	1.58
4	0.00	0.00	1.05	18.90	0.00	0.00	1.58	0.00

Policy Beginning in Year 2

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
2	0.00	82.69	0.00	0.00	19.85	0.00	9.92	9.92
3	110.25	22.05	74.97	19.85	13.23	33.08	14.88	8.27
4	0.00	5.51	3.31	39.69	0.00	0.00	6.62	1.65

Policy Beginning in Year 3

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
3	0.00	86.82	0.00	0.00	20.84	0.00	10.42	10.42
4	115.76	23.15	78.72	20.84	13.89	34.73	15.63	8.68

Policy Beginning in Year 4

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
4	0.00	91.16	0.00	0.00	21.88	0.00	10.94	10.94

Sum all entries for all policies in the indicated years.

Total

	Premium		Losses		Expenses			
Year	**Earned**	**Received**	**Incurred**	**Paid**	**STAT Incurred**	**GAAP Incurred**	**Paid Expenses**	**STAT Reserve**
0	0.00	75.00	0.00	0.00	18.00	0.00	9.00	9.00
1	100.00	98.75	68.00	18.00	30.90	30.00	22.95	16.95
2	105.00	108.69	74.40	54.90	32.45	31.50	30.10	19.30
3	110.25	114.12	79.12	75.65	34.07	33.08	33.10	20.26
4	115.76	119.83	83.08	79.43	35.77	34.73	34.76	21.28

b. ROE is end-of-year income / beginning-of-year equity

Year	Reserves[1]	Surplus[2]	Premiums Receivable	Invested Assets[3]	Underwriting Income[4]	GAAP Equity[5]
0	109.00	20.20	25	104.20	0	38.20
1	171.95	36.95	31.25	177.65	2	55.85
2	199.05	44.15	32.81	210.38	-0.90	63.99
3	209.00	46.35	34.45	220.90	-1.95	67.19
4	219.45	48.67	36.18	231.95	-2.04	70.55

Year	After-Tax Income[6]	**ROE[7]**
0	0	**n/a**
1	5.36	**14.04%**
2	6.34	**11.36%**
3	6.94	**10.85%**
4	7.29	**10.85%**

[1] Reserves in policy year 0 are 100 in unearned premium + 9 in stat reserve; thereafter reserves = cumulative incurred – cumulative paid.

For the first policy:

$$\text{Reserves}_0 = 109$$

$$\text{Reserves}_1 = (68 - 18) + 7.5 = 57.5$$

$$\text{Reserves}_2 = [(68 + 3) - (18 + 36)] + 1.5 = 18.5$$

$$\text{Reserves}_3 = 0$$

For each policy thereafter, we multiply by 1.05

So total reserves in Year 0 = 109

Reserves in Year 1 = $57.5 + 109(1.05) = 171.95$

Reserves in year 2 = $18.5 + 57.5(1.05) + 109(1.05)^2 = 199.05$

Reserves in year 3 = $0 + 18.5(1.05) + 57.5(1.05)^2 + 109(1.05)^3 = 209.00$

Reserves in year 4 = $0 + 0 + 18.5(1.05)^2 + 57.5(1.05)^3 + 109(1.05)^4 = 219.45$

[2] Surplus is based on discounted unpaid losses. The surplus pattern for the first policy is

20.20, 15.74, 5.35, 0, 0

Sample Calc for Year 1 surplus: $S_1 = 31.5\% \left(\frac{36}{1.06} + \frac{18}{1.06^2}\right) = 15.74$

Surplus in Year 0: 20.20

Surplus in Year 1: 15.74 + 20.20(1.05) = 36.95, and so forth

[3] Invested assets are reserves + surplus – premium receivable

Premium Receivable for the first policy follows the pattern (starting at year 0): 25, 5, 0, 0, 0

Receivables in year 0 = 25

Receivables in year 1 = 5 + 25(1.05) = 31.25

Receivables in year 2 = 0 + 5(1.05) + 25(1.05)^2 = 32.81, and so forth

[4] Underwriting Income is earned premium – incurred losses – GAAP incurred expenses

[5] GAAP Equity = Surplus, except in the first policy year, which also includes DAC (year 0 expenses in STAT)

For the first policy, GAAP Equity = 38.20, 15.74, 5.35, 0, 0

Year 0: 38.20

Year 1: 15.74 + 38.20(1.05) = 55.85

Year 2: 5.35 + 15.74(1.05) + 38.20(1.05)^2 = 63.99

[6] After-Tax Income – For year 1: $(1 - 35\%)\big(2 + 6\%(104.20)\big) = 5.36$

[7] ROE for year 1 - $5.36 \div 38.20 = 14.04\%$

Rob-34 Sol. This is just a random list of assumptions. The only thing that matters is that the growth rate = IRR = 9%, so **ROE = 9%.**

Kreps (Riskiness Leverage Models)

Kre-1 Sol. An allocatable risk load formulation should be allocatable to any level of definition; should be additive; should be appropriate to use for any subgroup or group of groups. In this way, senior management can allocate to regions, regional managers can allocate to lines of business, and line of business allocations will sum back to the original.

Kre-2 Sol. Coherent measures exhibit subadditivity, positive homogeneity, translational invariance, and monotonicity. The subadditivity measure may or may not be satisfied for a given leverage ratio.

Kre-3 Sol.

a. Coherence – Constant, TV@R

b. Entire distribution – Variance

c. Downside deviations – Proportional Excess, Mean Downside Deviation, semi-variance, V@R, TV@R

d. Risk of meeting plan – Constant, semi-variance, mean downside deviation

Kre-4 Sol. Since $R = \int f(x)\,(x - \mu)L(x)dx$, there is already a currency dimension in the product with $L(x)$ (e.g., the output of $\int f(x)\,(x - \mu)dx$ would be dollars, if x were measured in dollars).

If $L(x)$ were not invariant to currency, the integral would no longer be able to be interpreted as a currency unit. We can construct $L(x)$ to satisfy this homogeneity by taking it as a ratio of currencies, such as x/μ, x/σ, or x/S (loss as a ratio of mean, standard deviation, or surplus).

Kre-5 Sol. Management is concerned with downside measures; desires constant leverage if excess is relatively small, increasing leverage as excess becomes relatively larger, and then decreasing or 0 after a point.

For regulators, leverage ratios should be zero until capital is seriously impacted, and should not decrease thereafter.

Kre-6 Sol.

a. Because Line S has a relatively higher magnitude of negative outcomes in the tail than do the other lines, it will require more capital using this measure. Thus the mean return on surplus will likely be relatively lower.

b. If Line S is written in conjunction with another policy from which it cannot be separated (e.g., homeowners' property and liability), it would not be possible to reduce volume of this business on its own. Also, regulatory requirements may make a reduction or exit unfeasible. Further, it can be logistically difficult to make a major change to portfolio composition.

Management may consider purchasing reinsurance to provide some surplus relief for Line S.

Kre-7 Sol.

a. Line B is riskier as it generates approximately 85% of the TV@R required in each scenario, as compared with Line A's roughly 13%. Line B is approximately **6 times** as risky.

b. For the worst 2% of scenarios, the average loss is $6.16MM. 1.5 times that is about **$9MM.**

c. To reduce surplus, management can write proportionately less business in Line B, or hedge the losses with a reinsurance policy.

Kre-8 Sol. Note that the mean loss size is:

$$0.9 \times 0 + 0.08 \times 10 + 0.02 \times 50 = 1.8$$

Also note that the formulation for risk load is $R = \int f(x)\,(x-\mu)L(x)dx$. For a discrete function, this is equivalent to $\sum L(x)f(x)(x-\mu)$.

a. For the constant function, $L(x) = c$

$$c\big(0.9(0-1.8) + 0.08(10-1.8) + 0.02(50-1.8)\big) = \mathbf{0}$$

For the constant function, the risk load is always 0.

b. For the variance function, $L(X) = \frac{\beta}{S}(x-\mu)$, where β is some constant and S is the company surplus.

$$\sum L(x)f(x)(x-\mu) = \sum \frac{\beta}{S}(x-\mu)f(x)(x-\mu)$$

$$\frac{\beta}{S}\sum (x-\mu)^2 f(x)$$

$$\frac{\beta}{S}(0.9(0-1.8)^2 + 0.08(10-1.8)^2 + 0.02(50-1.8)^2)$$

$$\frac{\beta}{S}Var[X] = \mathbf{54.76\frac{\beta}{S}}$$

In this case, if we let the risk load equal to surplus, $S = \frac{\beta}{S}Var[X] \rightarrow S = \sqrt{\beta Var[X]}$. *For the variance function, the risk load is a multiple of the standard deviation of the aggregate loss distribution.*

c. For the TV@R function, $L(X) = \frac{\theta(x-x_q)}{1-q}$. In this case $q = 0.95$, so $x_q = 10$.

$$\sum L(x)f(x)(x-\mu) = \sum \frac{\theta(x-x_q)}{1-q} f(x)(x-\mu)$$

Since $q = 0.95$, consider only the last 5% of the distribution

$$0.03\left[\frac{1}{1-0.95}\right][10-1.8] + 0.02\left[\frac{1}{1-0.95}\right][50-1.8]$$

$$\mathbf{24.2}$$

Note that the total capital in this case would be $C = \mu + R = 1.8 + 24.2 = 26$.

26 is the TV@R if you calculate directly as the weighted average of the worst 5% of scenarios.

Note: Here I have used $\theta(0) = 1$, largely because this makes it consistent with the definition of TV@R. As noted in the summary, value of θ at 0 is not well-defined. This would not make for a particularly good exam question because of the ambiguity, although it is always good form to make sure your results checks out with the "conventional calculation."

Per a suggestion from a student, I added the next question which would remove the need to assign a value to $\theta(0)$.

d. With $x_q = 0$ when $q = 0.90$, we consider last 10% of distribution:

$$0.08\left[\frac{1}{1-0.90}\right][10-1.8] + 0.02\left[\frac{1}{1-0.90}\right][50-1.8] = \mathbf{16.2}$$

Again note that total capital of 16.2 + 1.8 = 18 aligns with the weighted average of the worst 10% of scenarios.

e. For the V@R function, $L(X) = \frac{\delta(x-x_q)}{f(x_q)}$. Again using $q = 0.95$, the only value that produces a non-zero delta function is when $x_q = 10$.

$$\sum \frac{\delta(x-10)}{f(x_q)} f(x)(x-\mu) = \frac{1}{.08}(.08)(10-1.8) = 8.2$$

Note that the total capital in this case would be $C = \mu + R = 1.8 + 8.2 = 10$, which is V@R(95%)

f. For the semi-variance function, $L(X) = \frac{\beta}{S}(x-\mu)\theta(x-\mu)$

$$\sum \frac{\beta}{S}(x-\mu)f(x)(x-\mu)\theta(x-\mu) = \frac{\beta}{S}\sum_{x>\mu}(x-\mu)^2 f(x)$$

$$\frac{\beta}{S}(0.08(10-1.8)^2 + 0.02(50-1.8)^2) = 51.844\frac{\beta}{S}$$

g. For mean downside deviation, $L(X) = \beta \cdot \frac{\theta(x-\mu)}{1-F(\mu)}$

Use of $\theta(x-\mu)$ indicates we are concerned only with values where $x > \mu$

$$\sum \beta \cdot \frac{\theta(x-\mu)}{1-F(\mu)} f(x)(x-\mu) = \frac{\beta}{1-F(\mu)} \sum_{x>\mu} (x-\mu) f(x)$$

$$\frac{\beta}{1-0.9}\big(0.08(10-1.8) + 0.02(50-1.8)\big) = 16.2\beta$$

Note that 16.2 = XTV@R(q | x_q = 1.8).

Mango (An Application of Game Theory: Property Catastrophe Risk Load)

Man-1 Sol. Using these methods, the order in which we add portfolios affects the outcome. For example, if Portfolio B is added to Portfolio A, it will be allocated a different risk load than it would be if Portfolio A were added to Portfolio B. This is undesirable because it produces mathematically ambiguous results.

Man-2 Sol. The marginal surplus method is sub-additive, so that, for two portfolios A and B, $MS(A) + MS(B) \leq MS(A + B)$. This is because the marginal surplus method utilizes standard deviation, which is itself sub-additive. (Perfect equality will happen only when the portfolios are perfectly positively correlated.)

Man-3 Sol. The marginal variance method is super-additive, so for two portfolios A and B, $MV(A) + MV(B) \geq MV(A + B)$. This is because the sum of the variances (left side) includes the covariance component twice.

Man-4 Sol. Renewal additivity means that the risk load needed for the combined portfolios X and Y is the same as the total risk load needed for X plus the additional risk load needed when Y is added to X, and vice versa. This is desirable so that we don't undervalue or overvalue the total risk load needed for a set of portfolios.

Man-5 Sol. The Shapley Value Method is great because regardless of the order in which portfolios are added, the risk load assigned to each portfolio does not change. Also, it adds up exactly to the total risk load. The Covariance Share Method is an improvement in that it generalizes the Shapley Value Method to assign the amount of covariance to a portfolio in proportion to its contribution to the covariance, where the Shapley method just uses an evenly-weighted split. This is preferred because intuitively events that contribute less to covariance should be assigned less capital.

Man-6 Sol. Yes, but it would be an ugly formula. The authors chose not to present it because it is not easily simplifiable, and it wouldn't result in anything much different than the much easier marginal variance approach.

Man-7 Sol.

- The surplus level requires $z = 1.96$.
- Management's required return implies $y = 12\%$.

a. $\frac{yz}{1+y}[\sigma_X] = \frac{(0.12)(1.96)}{1.12}\sqrt{200} = \mathbf{2.970}$

b. $\frac{yz}{1+y}[\sigma_Y] = \frac{(0.12)(1.96)}{1.12}\sqrt{50{,}000} = \mathbf{46.957.}$

c. $\frac{yz}{1+y}[\sigma_{X+Y}] = \frac{(0.12)(1.96)}{1.12}\sqrt{52{,}000} = \mathbf{47.887.}$

d. $\frac{yz}{1+y}[\sigma_{X+Y} - \sigma_Y] = \frac{(0.12)(1.96)}{1.12}\left(\sqrt{52{,}000} - \sqrt{50{,}000}\right) = \mathbf{0.930}.$

e. $\frac{yz}{1+y}[\sigma_{X+Y} - \sigma_X] = \frac{(0.12)(1.96)}{1.12}\left(\sqrt{52{,}000} - \sqrt{200}\right) = \mathbf{44.918}.$

f. The marginal surplus method produces a <u>sub-additive result</u>. The marginal sum is 2.04 less than the sum of the total business.

Man-8 Sol.

a. $\frac{yz}{1+y} \div \sigma_{X+Y}\left[Var[X]\right] = \frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [200] = \mathbf{0.184}$

b. $\frac{yz}{1+y} \div \sigma_{X+Y}\left[Var[Y]\right] = \frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [50{,}000] = \mathbf{46.046}.$

c. $\frac{yz}{1+y} \div \sigma_{X+Y}\left[Var[X+Y]\right] = \frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [52{,}000] = \mathbf{47.887}.$

d. $\frac{yz}{1+y} \div \sigma_{X+Y}\left[Var[X+Y] - Var[Y]\right] = \frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [52{,}000 - 50{,}000] = \mathbf{1.842}.$

e. $\frac{yz}{1+y} \div \sigma_{X+Y}\left[Var[X+Y] - Var[X]\right] = \frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [52{,}000 - 200] = \mathbf{47.703.}$

f. The marginal variance method produces a <u>super-additive result</u>. The marginal sum is 1.66 greater than the sum of the total business.

Man-9 Sol.

a. Same as in marginal variance method. **0.184.**

b. Likewise, we would have **46.046.**

c. 47.887

d. Since Var(X+Y) = Var(X) + Var(Y) + 2Cov(X,Y) we have Cov(X,Y) = 900. We add this to each marginal variance risk load. The risk load for X, if added to Y, would be:

$$\frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [200 + 900] = \mathbf{1.013}$$

e. $\frac{(0.12)(1.96)}{1.12} \div \sqrt{52{,}000} \times [50000 + 900] = \mathbf{46.874}.$

f. The sum of (d) and (e) is the same **47.887.**

Man-10 Sol. We expand the table a bit to determine covariance and each line's share.

Event	Probability	LOB A	LOB B	Cov(A, B)	CS_A	CS_B
1	0.2%	20	130	5.19	1.38	9.00
2	0.5%	50	90	22.39	15.99	28.78
3	3.0%	70	120	244.44	180.11	308.77
Total				272.02	197.49	346.55

Reasonability Check: The total covariance share is twice the covariance.

Since Var(A+B) = Var(A) + Var(B) + 2Cov(A,B),

$$1192.93 = Var[A] + 493.07 + 2(272.02)$$
$$Var[A] = 155.83$$

Sample Calcs

- $CV_{A+B,3} = (3\%)(1-3\%)(70)(120) = 244.44$
- $CS_{A,3} = 2\left(\frac{70}{120+70}\right)(244.44) = 180.11$
- $CS_{B,1} = 2\left(\frac{130}{20+130}\right)(5.19) = 9.00$

Method	Line A	Line B
Marginal Surplus	$0.14 \cdot \frac{2.0}{1.14}\left(\sqrt{1{,}192.93} - \sqrt{493.07}\right)$ **3.029**	$0.14 \cdot \frac{2.0}{1.14}\left(\sqrt{1{,}192.93} - \sqrt{155.83}\right)$ **5.417**
Marginal Variance	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(1{,}192.93 - 493.07)$ **4.977**	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(1{,}192.93 - 155.83)$ **7.375**
Shapley Method	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(155.83 + 272.02)$ **3.042**	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(493.07 + 272.02)$ **5.441**
Covariance Share	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(155.83 + 197.49)$ **2.513**	$0.14 \cdot \frac{2.0}{1.14 \cdot \sqrt{1{,}192.93}}(493.07 + 346.55)$ **5.971**

Reasonability Check for renewals:

$$MS_A + MS_B < Shapley_A + Shapley_B = CS_A + CS_B < MV_A + MV_B$$

Man-11 Sol. A super-additive characteristic function in one in which "the whole is greater than the sum of its parts." In other words, the utility of a combined group is better than the sum of the utility of the individuals in the group. A cooperative game would naturally tend toward a super-additive characteristic function. The optimal number of coalitions is one group (1); everyone should merge to optimize outcomes. (This big group is called the "grand coalition.")

Man-12 Sol. Since the game is super-additive, coalitions will always be better off (or at least not worse) by merging. There should be one coalition of all N players.

Man-13 Sol.

a. It is super-additive. Alone, no one can purchase anything, but together they have $10, and could purchase 750 beads and allocate amongst themselves.

b. While ABC are all better in the coalition that C proposed than they would be alone, AB would do better to leave the coalition – they could pool their money, purchase 500 beads, and then split evenly. C's proposal does not have merit because AB are better off without C in this situation – the coalition is not stable.

c. Yes – Anna's proposal is stable. All 750 beads are allocated, and no one would be better off on their own or in a sub-coalition. Anna obviously is much better off; Belle and Carrie are no worse than before, nor can they improve by banding together.

Man-14 Sol. **(★★)** This method of allocation is the Shapley method. Consider all the possible orderings.

Solution #1 (Binary Allocation)

This method assigns credit binarily – either the monsters all win together or they lose together; whoever changes that decision gets credit. It is akin to assuming that the monsters only care about whether they'll lose HP in vain or not, so will award points to the first monster who turns their sure loss into a sure win.

I wrote them out here for completeness, but note that B, C, and D provide the same utility for purposes of conquering the dragon.

Ordering	A	B	C	D
ABCD or ABDC	0	30,000	0	0
ACBD or ACDB	0	0	30,000	0
ADBC or ADCB	0	0	0	30,000
A second (6 combinations)	30,000	0	0	0
A third (6 combinations)	30,000	0	0	0
BCDA or CBDA	0	0	0	30,000
DBCA or BDCA	0	0	30,000	0
CDBA or DCBA	0	30,000	0	0

In case the table is unclear: The first row means we add A, then B, then C, then D (or C then D). With A alone, the monsters cannot win the prize. Once B is added, there is enough to win, so B's marginal contribution is all 30,000 XP. After that, C and D don't add any more value.

Average Marginal Contribution

A: $\frac{12(30{,}000)}{24} = 15{,}000$

B, C, D: $\frac{4(30{,}000)}{24} = 5{,}000$

Note that the allocation is additive, as we expect for Shapley method.

Solution #2 (Allocate Incrementally)

This method assumes that the monsters fight sequentially, so one monster fully exhausts his HP before the next monster enters. The next monster gives the lesser of the remaining HP left to defeat Eternatus and the monster's total HP.

See the Excel file for the full set of scenarios, but a sample few look as below:

Order	1	2	3	A	B	C	D
ABCD	450	80	0	45/53	8/53	0	0
ABDC	450	80	0	45/53	8/53	0	0
BACD	250	280	0	28/53	25/53	0	0
BADC	250	280	0	28/53	25/53	0	0
BCAD	250	150	130	13/53	25/53	15/53	0
BCDA	250	150	130	0	25/53	15/53	13/53

The green cells indicate that the monster who contributed 1, 2, or 3rd has contributed sufficiently to defeat the monster. The portions on the right part of the table indicate the portion of points awarded to the monster in each scenario, based on the monster's points seen in his order position.

The average value of points earned by each monster in the twenty-four possible scenarios work out to be:

	Portion	**Points**
A	24/53	**13,585**
B	13/53	**7,358**
C	8/53	**4,528**
D	8/53	**4,528**
Total	1	**30,000**

Note that the allocation is additive, as we expect for Shapley method.

Man-15 Sol. Assuming the insurers are virtually identical, A's proposal **not fair**, since A sees a bigger reduction in its capital contribution than the other would.

If the three insurers combined, assuming another split is not proposed, the insurers would presumably build off of A's scenario, so that the contributions would be:

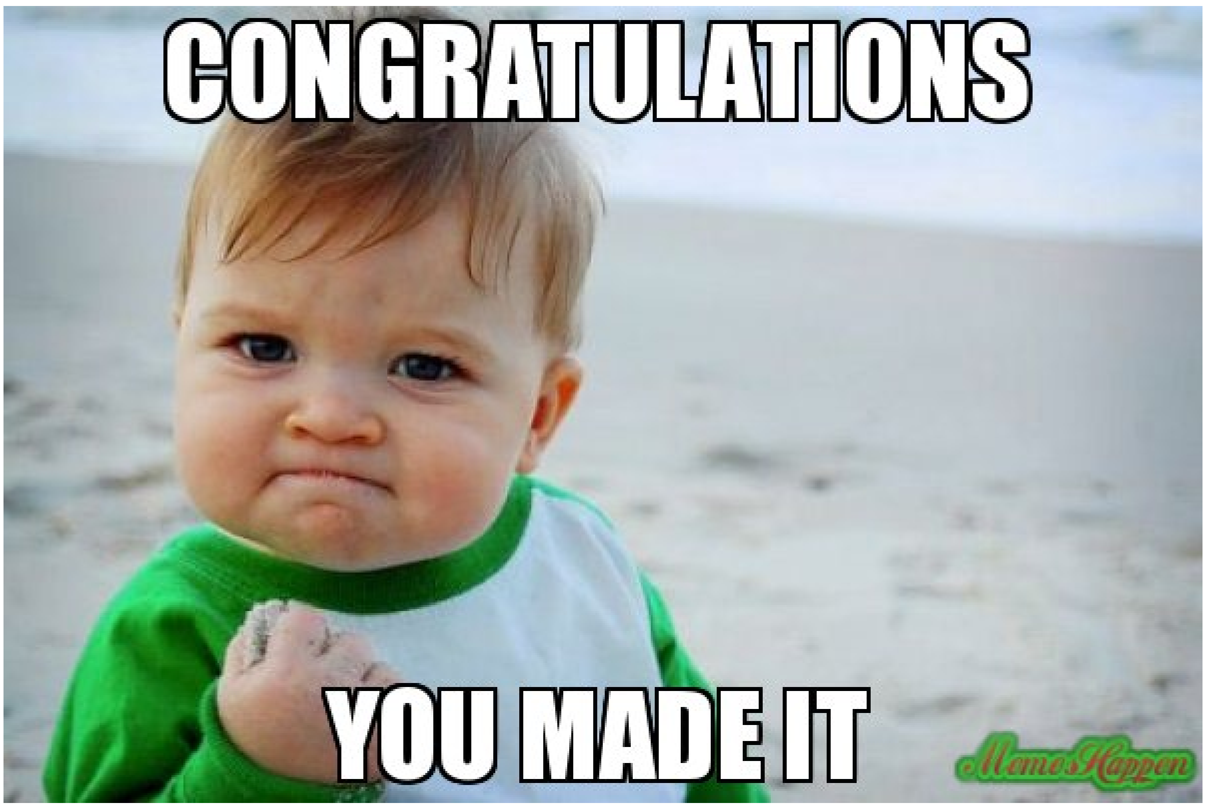
A: 750   Second Insurer: 1,700 – 750 = 950   Third Insurer: 2,500 – (750 + 950) = 800

This **does meet the conditions for individual and collective rationality.** Each insurer is better off in the coalition than outside of it. Since there was no alternative combination scenario proposed for the subgroups, then no insurer is better off forming its own subgroup. (This is trivially true for the smaller potential subgroup – A and the second insurer are exactly in the same position as they would have been in their own subgroup, but they are not worse.)

Man-16 Sol. *This solution is in Excel file 9-7, with the ability to change the inputs.*

- Alone, Entity 2 pays 3.2 million.
- If it were to partner with E1, E1 would pay at most 1.9, so E2 pays at least $4.5 - 1.9 = 2.6$
- Likewise, partnering with E3 means E2 pays at least $7.5 - 5.1 = 2.4$
- If the full coalition is formed, E1 and E3 combined would pay no more than they could in a group of their own, or 6.4; therefore, E2 must pay at least $8.7 - 6.4 = 2.3$ in this group.
- The range of funds satisfying individual and rational collectivity is the minimum and maximum values shown above, or **$2.3 to 3.2 million.**

😊 End of Learning Objective D 😊



Good luck on your exam! Let me know if you have questions ☺